Functional analytic approach to non-local self-improving properties

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Plan of the talk

- Elliptic equations and higher integrability of gradient. Meyers' estimate.
- Non-local equations and higher differentiability. Kuusi–Mingione–Sire theorem.
- A functional analytic approach.
- Evolutionary variant.

Part I: Elliptic equations and higher integrability of gradient.

Gehring's lemma

Lemma (Gehring)

Let $p > 1$ be fixed and let $w, f \geq 0$ be locally integrable functions satisfying

$$\left(\fint_B w^p \right)^{1/p} \leq C \fint_{2B} w + \fint_{2B} f$$

for all balls B. Then there is $\epsilon > 0$ such that for all balls B

$$\left(\fint_B w^{p+\epsilon} \right)^{1/(p+\epsilon)} \leq C \fint_{2B} w + C \left(\fint_{2B} f^{p+\epsilon} \right)^{1/(p+\epsilon)}$$

- This is the open-ended property of reverse Hölder classes ($f = 0$).
- When $p \to 1$, the analogous result is the inclusion $A_\infty \subset \mathrm{RHI}_{1+\epsilon}$.

Consider the following set-up:

- $A : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is a measurable map into real $n \times n$ matrixes satisfying $\lambda|\xi|^2 \leq \xi \cdot A(x)\xi$ and $|A(x)| \leq \Lambda$ for fixed constants $\lambda, \Lambda \in (0, \infty)$ and all $x \in \mathbb{R}^n$.
- $f \in L^{2+\epsilon_1}_{loc}(\mathbb{R}^n)$ is a real valued source term.
- $u \in W^{1,2}(\mathbb{R}^n)$ is a weak solution to

$$-\operatorname{div}(A\nabla u) = \operatorname{div} f.$$

Theorem (Meyers' estimate (1963))

The solution u satifies $u \in W^{1,2+\epsilon_2}_{loc}(\mathbb{R}^n)$ for some $\epsilon_2 \in (0, \epsilon_1)$.

Remarks:

- A priori, one only assumed $W^{1,2}(\mathbb{R}^n)$. There is improvement in local integrability.
- Meyers' estimate builds on earlier work by Bojarski (the planar case, systems). It is also valid for complex equations.
- Systems (in all dimensions) were treated by Elcrat and Meyers.
- And so on.

The usual proof

- a Caccioppoli estimate:

$$\fint_{B_r} |\nabla u|^2 \lesssim r^{-2} \fint_{B_{2r}} |u|^2 + \fint_{B_{2r}} f^2$$

(weak formulation with test function $u\varphi^2$ where the smooth bump satisfies $1_{B_r} \leq \varphi \leq 1_{B_{2r}}$)

- the Sobolev-Poincaré inequality:

$$r^{-1} \left(\fint_{B_{2r}} |u - u_{B_{2r}}|^2 \right)^{1/2} \lesssim \left(\fint_{B_{3r}} |\nabla u|^{2_*} \right)^{1/2_*}$$

where $1/n = 1/2_* - 1/2$.

- interpolation of L^p norms (Hölder) to lower the exponent on the right.
- Gehring's lemma to win an epsilon in the exponent on the left.

Part II: Non-local equations

Smoothness of order one

Consider the functional

$$\mathcal{F}_1(u) = \int_{\mathbb{R}^n} \nabla u(x) \cdot A(x) \nabla u(x)\, dx$$

with natural domain $W^{1,2}(\mathbb{R}^n)$. Morally, this is perturbation of $\int |\nabla u|^2\, dx$

Equation $-\operatorname{div}(A\nabla u) = f$, that is,

$$\int_{\mathbb{R}^n} A(x)\nabla u(x) \cdot \nabla\varphi(x)\, dx = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx, \quad \forall \varphi \in C_c^\infty$$

means the first variation of $\mathcal{F}_1(u) - \int fu$ vanishing.

The natural domain of $-\operatorname{div}(A\nabla \cdot)$ is again $W^{1,2}(\mathbb{R}^n)$ and its range is the dual space $W^{1,2}(\mathbb{R}^n)^*$.

Fractional Sobolev spaces

Consider the following seminorm

$$|u|_{W^{s,p}(\mathbb{R}^n)} = \iint \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy, \quad 0<s<1, \quad p>1$$

and the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + |u|_{W^{s,p}(\mathbb{R}^n)}$$

- Define: $u \in W^{s,p}(\mathbb{R}^n)$ if $u \in L^p$ and $\|u\|_{W^{s,p}(\mathbb{R}^n)} < \infty$.
- In systematical study of function spaces $W^{s,p} = B^s_{p,p}$.
- Not to be confused with the Bessel potential spaces $H^{s,p} = F^s_{p,2}$ that only coincide with $W^{s,p}$ when $p=2$.

Fractional equations

Consider then

$$\mathcal{F}(u) = \iint A(x,y) \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} \, dxdy, \quad 0 < s < 1$$

and its first variation

$$\mathcal{E}_s(u,\varphi) := \iint A(x,y) \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}$$

(assume $\lambda \leq |A(x,y)| \leq \lambda^{-1}$ with $\lambda \in (0,1)$).

- The natural domain for the functional is $\dot{W}^{s,p}(\mathbb{R}^n)$.
- Its derivative is a functional defined through the form $\mathcal{E}_s$. Not clear where.

Core part of the non-local self-improving theorem

Theorem (Kuusi–Mingione–Sire (2015))

Denote $2_* = 2n/(n+2s)$, *let* $\delta_0 > 0$, *take* $f \in L^{2_*+\delta_0}_{loc}(\mathbb{R}^n)$. *Suppose that* $u \in W^{s,2}(\mathbb{R}^n)$ *is a solution in the sense that for all* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *it holds*

$$\iint A(x,y)\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dxdy = \int f\varphi\,dx.$$

Then $u \in W^{s+\delta_1,2+\delta_1}_{loc}(\mathbb{R}^n)$ *for some* $\delta_1 > 0$.

Remark. There is gain in both integrability and differentiability. The right hand side can be more general, but this will be discussed later.

Some words on KMS's proof

- High level: Establish a reverse Hölder inequality for a suitable quantity and prove an appropriate Gehring's lemma.
- **Dual pairs**: write

$$|u|_{W^{s,p}(\mathbb{R}^n)} = \iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} = \int_{\mathbb{R}^n\times\mathbb{R}^n} |U(x,y)|^2 \, d\mu(x,y)$$

  where $U(x,y) = (u(x)-u(y))/|x-y|^{s+\epsilon}$, $d\mu = |x-y|^{n-2\epsilon}$.
- **Mixing property**: The extension $U(x,y)$ mixes the smoothness source (order of finite difference in u, qualitative in exponent) and drain (division by $|x-y|$, quantitative in exponent) conveniently. Higher integrability in the spirit of Gehring gives control over more smoothness.

Remarks

- Kuusi–Mingione–Sire worked with a slightly more general non-linear operator.
- Higher integrability for fractional equations was done earlier by Bass–Ren (2013).
- The higher differentiability was extended to p-fractional setting (related to $W^{s,p}(\mathbb{R}^n)$) by Schikorra (2016) with a simpler proof.
- The rest of the talk will be about the functional analytic approach (joint work with Pascal Auscher, Simon Bortz and Moritz Egert).

Part III: Functional analytic approach

Elementary properties for the proof

Recall the form (now with complex valued functions, $\lambda < \operatorname{Re} A(x,y) \leq |A(x,y)| \leq \lambda^{-1}$)

$$\mathcal{E}_s(u,\varphi) := \iint A(x,y) \frac{(u(x)-u(y))\overline{(\varphi(x)-\varphi(y))}}{|x-y|^{n+2s}}.$$

- Boundedness: $|\mathcal{E}_s(u,\varphi)| \leq C|u|_{W^{s,2}}|\varphi|_{W^{s,2}}$ by Hölder. Define $\mathcal{L} : W^{s,2} \to (W^{s,2})^*$ via $\langle \mathcal{L}u, \varphi \rangle = \mathcal{E}_s(u,\varphi)$.
- $\mathcal{E}_s$ is quasi-coercive on $W^{s,2}$, that is, $\operatorname{Re} \mathcal{E}_s(u,u) \geq \lambda |u|_{W^{s,p}}$.
- By Lax-Milgram lemma $1+\mathcal{L} : W^{s,2} \to (W^{s,2})^*$ is invertible, $\max(\|1+\mathcal{L}\|, \|(1+\mathcal{L})^{-1}\|) \leq \lambda^{-1}+1$.

Alternative domains of the form

On the other hand

$$\mathcal{E}_s(u,\varphi) = \iint A(x,y)\frac{(u(x)-u(y))\overline{(\varphi(x)-\varphi(y))}}{|x-y|^{n+2s}}$$
$$= \iint A(x,y)\frac{(u(x)-u(y))}{|x-y|^{n/p+\alpha}}\frac{\overline{(\varphi(x)-\varphi(y))}}{|x-y|^{n/p'+\beta}}$$

$\mathcal{L} : W^{\alpha,p} \to (W^{\beta,p'})^*$ whenever $\alpha+\beta = 2s$ and $1/p + 1/p' = 1$.

These include the spaces near $W^{s,2}$. We will prove invertibility close enough. All conditions on the right hand side of the equation are just conditions to include it to dual of some space nearby.

Recall the complex interpolation

Let X, Y be Banach spaces contained in tempered distributions. Let $S = \{z \in \mathbb{C} : \operatorname{Re} z \in (0,1)\}$. We say $f \in \mathcal{F}(X,Y)$ if

- $f : S \to X + Y$ is holomorphic on S.
- $t \mapsto f(0+it)$ is continuous $\mathbb{R} \to X$ and $f(it) \to 0$ as $|t| \to \infty$.
- $t \mapsto f(1+it)$ is continuous $\mathbb{R} \to Y$ and $f(1+it) \to 0$ as $|t| \to \infty$.

Let $\|f\|_{\mathcal{F}(X,Y)} = \max(\sup_t \|f(it)\|_X, \sup_t \|f(1+it)\|_Y)$.

- Set $[X,Y]_{[\theta]} = \{f(\theta) : f \in \mathcal{F}(X,Y)\}$.
- Define the norm $\|u\|_{[X,Y]_{[\theta]}} = \inf\{\|f\|_{\mathcal{F}(X,Y)} : f(\theta) = u\}$
- $[X,Y]_\theta$ is the complex interpolation space of X and Y.

Small perturbations

Theorem (Shneiberg (1974))

Let (X_0, X_1) *and* (Y_0, Y_1) *be two interpolation couples and* T *a bounded linear operator* $X_0 \to Y_0$ *and* $X_1 \to Y_1$. *Assume that* $\theta_0 \in (0,1)$ *such that* T *is invertible* $X_{\theta_0} \to Y_{\theta_0}$.

Then there is $\delta > 0$ *only depending on the data above so that*

$$T : X_\theta \to Y_\theta \textit{ is invertible for all } \theta \in (\theta_0 - \delta, \theta_0 + \delta)$$

and the inverses agree on $X_\theta \cap X_{\theta_0}$.

Complex interpolation of fractional Sobolev spaces

Proposition (from a textbook, e.g. Bergh and Löfström 1976)

Let $s_0, s_1 \in (0,1)$ *and* $1 < p_0, p_1 < \infty$. *Set* $\theta \in (0,1)$ *and*

$$s = \theta s_0 + (1-\theta)s_1, \qquad \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$$

then

$$[W^{s_0,p_0}, W^{s_1,p_1}]_\theta = W^{s,p}.$$

The dual spaces interpolate similarly.

Conclusion of the proof

By Shneiberg's theorem there is $\delta > 0$ such that $\mathcal{L} : W^{\alpha,p} \to (W^{\beta,p'})^*$ is invertible whenever $\alpha + \beta = 2s$, $1/p + 1/p' = 1$ and $(\alpha, 1/p) \subset B((s, 1/2), \delta)$.

For instance, if $f \in L^r$ with

$$\frac{2s - \alpha}{n} = \frac{1}{r} - \frac{1}{p}, \quad (s - \alpha), (p - 2) \in (0, \epsilon),$$

for $\epsilon > 0$ small enough and $u \in W^{s,2}$ is a solution to $\mathcal{L}u = f$, then $u \in W^{\alpha,p}$.

A few remarks

- Improvement in differentiability strongly depends on the structure of the form $\mathcal{E}$. There was a way to choose how much smoothness one requires from the solution and test function.
- The form $(u, \varphi) \mapsto \int A(x)\nabla u \cdot \nabla \varphi$ fixes smoothness because the non-smooth coefficients do not allow to rearrange derivatives.
- The form $(u, \varphi) \mapsto \int a(x)[(-\Delta)^{\alpha} u][(-\Delta)^{\alpha} \varphi]$ with $\alpha < 1/2$ is from a non-local equation (associated to potential spaces), but it does not allow for redistributing derivatives. Its solutions do not gain smoothness.

Part IV: One more application

Evolutionary variant

Let $A_t(x,y)$ satisfy again $\lambda < \operatorname{Re} A_t(x,y) \leq |A_t(x,y)| \leq \lambda^{-1})$ for all $t > 0$ and let $\mathcal{L}_{A_t} : W^{s,2}(\mathbb{R}^n) \to W^{s,2}(\mathbb{R}^n)^*$ be the operator defined through the form

$$\mathcal{E}_{s,A_t}(u,\varphi) := \iint A_t(x,y) \frac{(u(x)-u(y))\overline{(\varphi(x)-\varphi(y))}}{|x-y|^{n+2s}}.$$

Consider the equation

$$\partial_t u(t) + \mathcal{L}_{A_t} u(t) = f(t), \quad u(0) = 0$$

posed in $[0,T] \times \mathbb{R}^n$ with data $f \in L^2(\mathbb{R}^{1+n})$.

Weak solutions are found in $H^1(0,T; W^{s,2}(\mathbb{R}^n)^*) \cap L^2(0,T; W^{s,2}(\mathbb{R}^n))$.

Application towards maximal regularity

Theorem (joint with Auscher, Bortz and Egert (2017))

Let $f \in L^2(0,T;L^2(\mathbb{R}^n))$. Then there is $\epsilon > 0$ and $\sigma > s$ and $p > 2$ such that the unique weak solution to

$$\partial_t u(t) + \mathcal{L}_{A_t} u(t) = f(t), \quad u(0) = 0$$

satisfies

$$u \in H^1(0,T;W^{s-\epsilon,2}(\mathbb{R}^n)^*) \cap L^2(0,T;W^{s+\epsilon,2}(\mathbb{R}^n))$$

and

$$u \in W^{\frac{\sigma}{2s},p}(0,T;L^p(\mathbb{R}^n)) \cap L^p(0,T;W^{s,p}(\mathbb{R}^n)).$$