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Abstract: *In this paper Hilbert spaces are characterized among Banach spaces in terms of transitivity with respect to nicely behaved subgroups of the isometry group. For example, the following result is typical: If* X *is a real Banach space isomorphic to a Hilbert space and convex-transitive with respect to the isometric finite-dimensional perturbations of the identity, then* X *is already isometric to a Hilbert space.*

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1 Introduction

The expression 'special symmetries' in the title refers to suitable subgroups of $\mathcal{G}(\mathrm{X}) = \{T \colon \mathrm{X} \to \mathrm{X} |\ T \text{ isometric automorphism}\}$ where X is a real Banach space. We denote the closed unit ball of X by $\mathbf{B}_{\mathrm{X}}$ and the unit sphere by $\mathbf{S}_{\mathrm{X}}$. The orbit of $x \in \mathbf{S}_{\mathrm{X}}$ with respect to a family $\mathcal{F} \subset L(\mathrm{X})$ is given by $\mathcal{F}(x) = \{T(x) |\ T \in \mathcal{F}\}$. An inner product $(\cdot|\cdot) \colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ is said to be *invariant* with respect to $\mathcal{F}$ if $(T(x)|T(y)) = (x|y)$ for each $x, y \in \mathrm{X}$, $T \in \mathcal{F}$. The concept of an invariant inner product is an important tool applied frequently in this article. We say that X is *transitive*, *almost transitive* or *convex-transitive* with respect to $\mathcal{F}$ if $\mathcal{F}(x) = \mathbf{S}_{\mathrm{X}}$, $\overline{\mathcal{F}(x)} = \mathbf{S}_{\mathrm{X}}$ or $\overline{\mathrm{conv}}(\mathcal{F}(x)) = \mathbf{B}_{\mathrm{X}}$, respectively, for all $x \in \mathbf{S}_{\mathrm{X}}$. If $\mathcal{F} = \mathcal{G}(\mathrm{X})$ above, then we will omit mentioning it. This article can be regarded as a part of the field generated around the well-known open *Banach-Mazur rotation problem*, which asks whether each transitive separable Banach space is isometrically a Hilbert space. See [3] for an exposition of the topic.

In [5] F. Cabello Sánchez studied the subgroup

$$\mathcal{G}_F = \{T \in \mathcal{G}(\mathrm{X}) |\ \mathrm{Rank}(T - \mathrm{Id}) < \infty\}$$

consisting of the finite-dimensional perturbations of the identity. There a classical result appearing in [1, 10] is applied, namely, that each finite-dimensional Banach space admits an invariant inner product. This motivated the work in [5], where an elegant proof was presented for the following result:

Theorem 1.1. *If the norm of* X *is transitive with respect to* $\mathcal{G}_F$*, then* X *is isometric to a Hilbert space.*

Cabello raised the question whether this result can be extended to the almost transitive setting. It turns out here that the answer is affirmative under the additional assumption that X is isomorphic to a Hilbert space:

Theorem 1.2. *Let* X *be a Banach space isomorphic to a Hilbert space. Then* X *is convex-transitive with respect to* $\mathcal{G}_F$ *if and only if* X *is isometric to a Hilbert space.*

This paper is also motivated by the following problems posed in [4, 5]:

- Is an almost transitive Banach space isometric to a Hilbert space if it is isomorphic to one?
- Find ideals $J \subset L(\mathrm{X})$ (with $F \subset J$) for which Theorem 1.1 remains true if condition $T - \mathrm{Id} \in F$ is replaced by $T - \mathrm{Id} \in J$ (here F is the ideal of finite-rank operators).

Questions of this type are treated here, and we will also show that the existence of an invariant inner product on X is determined by the existence of invariant inner products separately with respect to finitely generated subgroups of $\mathcal{G}(\mathrm{X})$ (see Theorem 2.2).

1.1 Preliminaries

We refer to [3], [8], [9] and [12] for some background information. Recall that a norm $||\cdot||$ on X is *maximal* if $\mathcal{G}_{(\mathrm{X},||\cdot||)} \subset \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$ for an equivalent norm $|||\cdot|||$ implies that $\mathcal{G}_{(\mathrm{X},||\cdot||)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$. If X is convex-transitive, then the norm of X is maximal, see [6]. We denote by Aut(X) the group of isomorphisms $T\colon \mathrm{X} \to \mathrm{X}$.

Given a topological group G we denote by UCB(G) the space of uniformly continuous bounded functions on G. Here we consider the uniform structure Φ_G of G as being generated by a basis of entourages of diagonal having the form

$$W = \{(g,h) \in G \times G |\ gh^{-1},\ g^{-1}h \in V\}, \tag{1}$$

where V runs over a neighbourhood basis of e in G. The space UCB(G) is endowed with the $||\cdot||_\infty$-norm.

For the sake of convenience we will enumerate the following condition: Suppose that there is a positive functional $F \in \mathrm{UCB}(G)^*$, $||F|| = 1$, such that

$$F(f(\cdot g)) = F(f(\cdot)) \quad \text{for all } f \in \mathrm{UCB}(G),\ g \in G. \tag{2}$$

This type of condition can be viewed as a weaker version of amenability of G (see [11]). We note that the rotation group of L^p with the strong operator topology is extremely amenable for $1 \leq p < \infty$, see [8].

Recall that the product topology of X^{X} inherited by $L(\mathrm{X})$ is called the strong operator topology (SOT).

We often consider subgroups $\mathcal{G} \subset \mathcal{G}(\mathrm{X})$, which enjoy the following property:

($*$) Given $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \mathcal{G}$ and a finite-codimensional subspace $\mathrm{Z} \subset \mathrm{X}$ there exists a finite-codimensional subspace $\mathrm{Y} \subset \mathrm{Z}$ such that $T_1(\mathrm{Y}) = \cdots = T_n(\mathrm{Y}) = \mathrm{Y}$.

Clearly $\mathcal{G}_F$ is an example of a subgroup of $\mathcal{G}(\mathrm{X})$ satisfying ($*$).

It is easy to see that if H is a Hilbert space, then $\mathcal{G}_F \subset \mathcal{G}(\mathrm{H})$ is dense in $\mathcal{G}(\mathrm{H})$ in the topology of uniform convergence on compact sets. On the other hand, given a Banach space X the group $\mathcal{G}(\mathrm{X})$ is SOT-closed in Aut(X).

2 Results

Theorem 2.1. *Let* X *be a maximally normed Banach space, which is isomorphic to a Hilbert space. Suppose that* $\mathcal{G}(\mathrm{X})$ *endowed with the strong operator topology is amenable in the sense of condition* (2)*. Then* X *is isometrically isomorphic to a Hilbert space.*

Proof. We may assume without loss of generality that $(\mathrm{X}, ||\cdot||)$ and $(\mathrm{X}, |\cdot|)$ are isomorphic via the identical mapping, where $|\cdot|$ is a norm induced by an inner product $(\cdot|\cdot)$ on X. We denote by $\mathcal{G}(\mathrm{X}) = \mathcal{G}_{(\mathrm{X},||\cdot||)}$ and $\mathcal{G}_{(\mathrm{X},|\cdot|)}$ the corresponding rotation groups, and these are regarded with the strong operator topology. Recall that $\Phi_{\mathcal{G}(\mathrm{X})}$ is the natural uniformity given by the group $(\mathcal{G}(\mathrm{X}), \mathrm{SOT})$ applied to (1).

Observe that $T \mapsto (Tx|Ty)$ defines a $\Phi_{\mathcal{G}(\mathrm{X})}$-uniformly continuous map $\mathcal{G}(\mathrm{X}) \to \mathbb{R}$ for each $x, y \in \mathrm{X}$. Indeed, this map is obtained by composing the $\Phi_{\mathcal{G}(\mathrm{X})}$-$||\cdot||_{\mathrm{X}\oplus_2\mathrm{X}}$ uniformly continuous map $\mathcal{G}(\mathrm{X}) \to \mathrm{X} \oplus_2 \mathrm{X},\ T \mapsto (Tx, Ty)$ and the map $(Tx, Ty) \mapsto (Tx|Ty)$, which is $||\cdot||_{\mathrm{X}\oplus_2\mathrm{X}}$-uniformly continuous as $||\cdot|| \sim |\cdot|$. To check that $T \mapsto (Tx, Ty)$ is uniformly continuous, first consider a standard entourage

$$E = \{(x_1, y_1, x_2, y_2) \in \mathrm{X} \oplus_2 \mathrm{X} \times \mathrm{X} \oplus_2 \mathrm{X} : \ ||(x_1, y_1) - (x_2, y_2)||_{\mathrm{X}\oplus_2\mathrm{X}} < \epsilon\}$$

for some $\epsilon > 0$. The preimage of this is

$$\begin{array}{cl} & \{(R, S) \in \mathcal{G}(\mathrm{X}) \times \mathcal{G}(\mathrm{X}) : \ ||(Rx, Ry) - (Sx, Sy)||_{\mathrm{X}\oplus_2\mathrm{X}} < \epsilon\}, \\ \supset & \{(R, S) \in \mathcal{G}(\mathrm{X}) \times \mathcal{G}(\mathrm{X}) : \ ||Tx - Sx||, ||Ty - Sy|| < \frac{\epsilon}{2}\} \\ = & \{(R, S) \in \mathcal{G}(\mathrm{X}) \times \mathcal{G}(\mathrm{X}) : \ ||x - T^{-1}Sx||, ||y - T^{-1}Sy|| < \frac{\epsilon}{2}\}. \end{array}$$

Hence it suffices to pick $V = \{R \in \mathcal{G}(\mathrm{X}) : \ ||x - Rx||, ||y - Ry|| < \frac{\epsilon}{2}\}$ in (1) to find an entourage of $\Phi_{\mathcal{G}(\mathrm{X})}$ in the preimage of E. We obtain that $T \mapsto (Tx, Ty)$ is $\Phi_{\mathcal{G}(\mathrm{X})}$-uniformly continuous.

According to the assumptions there is $F \in \mathrm{UCB}(\mathcal{G}(\mathrm{X}))^*,\ ||F|| = 1$, such that $F(f(\cdot g)) = F(f(\cdot))$ for $f \in \mathrm{UCB}(\mathcal{G}(\mathrm{X}))$ and $g \in \mathcal{G}(\mathrm{X})$. For each $x, y \in \mathrm{X}$ we put

$$[x|y] = F(\{(g(x)|g(y))\}_{g\in\mathcal{G}(\mathrm{X})}).$$

This definition is sensible, since $g \mapsto (g(x)|g(y))$ defines an element in $\mathrm{UCB}(\mathcal{G}(\mathrm{X}))$ for each $x, y \in \mathrm{X}$. We claim that $[\cdot|\cdot]$ defines an inner product on X such that $|||x||| \doteq \sqrt{[x|x]}$ is equivalent to $||\cdot||$. Indeed, first note that $[\cdot|\cdot]\colon (\mathrm{X}, ||\cdot||) \oplus_2 (\mathrm{X}, ||\cdot||) \to \mathbb{R}$ is defined and bounded, since $(\cdot|\cdot)\colon (\mathrm{X}, ||\cdot||) \oplus_2 (\mathrm{X}, ||\cdot||) \to \mathbb{R}$ is bounded and $||F|| = 1$. By using the bilinearity of $(\cdot|\cdot)$ and the linearity of F we obtain that $[\cdot|\cdot]$ is bilinear. Let $C \geq 1$ such that $C^{-2}||\cdot||^2 \leq |\cdot|^2 \leq C^2||\cdot||^2$. Since F is positive and norm-1, we get that

$$C^{-2}||x||^2 = \inf_g C^{-2}||g(x)||^2 \leq F(\{(g(x)|g(x))\}_{g\in\mathcal{G}(\mathrm{X})}) \leq \sup_g C^2||g(x)|| = C^2||x||,$$

where $x \in \mathrm{X}$ and the supremum and infimum are taken over $\mathcal{G}(\mathrm{X})$. This means that $[\cdot|\cdot]$ is an inner product on X such that $|||\cdot|||$ is equivalent to $||\cdot||$.

Observe that

$$[h(x)|h(y)] = F(\{(gh(x)|gh(y))\}_{g\in\mathcal{G}(\mathrm{X})}) = F(\{(g(x)|g(y))\}_{g\in\mathcal{G}(\mathrm{X})}) = [x|y]$$

for each $h \in \mathcal{G}(\mathrm{X})$. The maximality of the norm of $(\mathrm{X}, ||\cdot||)$ yields that $\mathcal{G}_{(\mathrm{X},||\cdot||)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$. The proof is completed by a standard argument using the fact that $(\mathrm{X}, |||\cdot|||)$ is transitive. □

Suppose that X is a Banach space with two equivalent norms $||\cdot||$ and $|||\cdot|||$ such that the group $\mathcal{G}$ generated by $\mathcal{G}_{(\mathrm{X},||\cdot||)} \cup \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$ is operator norm bounded. Then there is one more equivalent norm $||||\cdot||||$ on X given by

$||||x|||| = \sup_{g\in\mathcal{G}} ||g(x)||$ and this is $\mathcal{G}$-invariant. Consequently, if the norms $||\cdot||$ and $|||\cdot|||$ are additionally maximal (resp. convex-transitive), then $\mathcal{G}_{(\mathrm{X},||\cdot||)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$ (resp. $||\cdot|| = c|||\cdot|||$ for some constant $c > 0$).

The argument employed in the proof of [5, Lemma 2] can be modified to obtain the following dichotomy regarding the existence of invariant inner products.

Theorem 2.2. *Let* X *be a Banach space and* $C \geq 1$. *Suppose that for each* $n \in \mathbb{N}$ *and* $T_1, \ldots, T_n \in \mathcal{G}(\mathrm{X})$ *there exists an inner product* $(\cdot|\cdot)_*\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}$ *invariant under the rotations* $T_1, \ldots, T_n$ *such that* $C^{-2}\ ||x||^2 \leq (x|x)_* \leq C^2\ ||x||^2$ *for each* $x \in \mathrm{X}$. *Then there is already an inner product* $(\cdot|\cdot)_{\mathrm{X}}\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}$, *which is invariant under* $\mathcal{G}(\mathrm{X})$ *and satisfies* $C^{-2}\ ||x||^2 \leq (x|x) \leq C^2\ ||x||^2$ *for* $x \in \mathrm{X}$.

Proof. We may assume without loss of generality that $\mathcal{G}(\mathrm{X})$ is not finitely generated. Let $\mathcal{N}$ be the net of finitely generated subgroups of $\mathcal{G}(\mathrm{X})$ ordered by inclusion. By the assumptions we may assign for each $\gamma \in \mathcal{N}$ an inner product $(\cdot|\cdot)_\gamma\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}$ invariant under γ and satisfying $C^{-1}||x||^2 \leq (x|x)_\gamma \leq C||x||^2$ for $x \in \mathrm{X}$. Observe that the sets $\{\gamma \in \mathcal{N}|\ \delta \subset \gamma\}$, where $\delta \in \mathcal{N}$, form a filter base of a filter $\mathcal{F}$ on $\mathcal{N}$. Let us extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ on $\mathcal{N}$. Note that $\mathcal{U}$ is non-principal, since for each $\eta \in \mathcal{N}$ there is $\delta \in \mathcal{N}$ with $\eta \subsetneq \delta$, so that $\eta \notin \{\gamma \in \mathcal{N}|\ \delta \subset \gamma\} \in \mathcal{U}$.

Define $B\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}^{\mathcal{N}}$ by setting $B(x,y) = \{(x|y)_\gamma\}_{\gamma\in\mathcal{N}}$ for $x,y \in \mathrm{X}$. We will consider $\mathbb{R}^{\mathcal{N}}$ equipped with the usual point-wise linear structure. Then B becomes a symmetric and bilinear map. Moreover, $B(x,x) \geq 0$ point-wise for $x \in \mathrm{X}$. Put $\vec{B}\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}$, $\vec{B}(x,y) = \lim_{\mathcal{U}} B(x,y)$ for $x,y \in \mathrm{X}$. Indeed, the above limit exists and is finite for all $x,y \in \mathrm{X}$, since $(x|y)_\gamma \leq \sqrt{(x|x)_\gamma (y|y)_\gamma} \leq C^2\ ||x||\ ||y||$ for all $\gamma \in \mathcal{N},\ x,y \in \mathrm{X}$. Moreover, similarly we get that $C^{-2}||x||^2 \leq \vec{B}(x,x) \leq C^2||x||^2$ for all $x \in \mathrm{X}$. It follows that $\vec{B}$ is an inner product on X.

Observe that for all $T \in \mathcal{G}(\mathrm{X})$ and $x,y \in \mathrm{X}$ we have that

$$\{\gamma \in \mathcal{N}|\ (Tx|Ty)_\gamma = (x|y)_\gamma\} \supset \{\gamma \in \mathcal{N}|\ T \in \gamma\} \in \mathcal{F} \subset \mathcal{U}.$$

Hence $\vec{B}(Tx,Ty) = \vec{B}(x,y)$ for $T \in \mathcal{G}(\mathrm{X})$ and $x,y \in \mathrm{X}$. Consequently, $\vec{B}$ is the required inner product. □

It is not known if an almost transitive Banach space isomorphic to a Hilbert space is in fact isometric to a Hilbert space (see [4]). The following consequence of Theorem 2.2 provides a partial answer to this problem.

Corollary 2.3. *Let* X *be a maximally normed Banach space,* H *a Hilbert space and* $C \geq 1$. *Suppose that for any* $n \in \mathbb{N}$ *and* $T_1, \ldots T_n \in \mathcal{G}(\mathrm{X})$ *there exists an isomorphism* $\phi\colon \mathrm{X} \to \mathrm{H}$ *such that* $\max(||\phi||, ||\phi^{-1}||) \leq C$ *and* $||\phi(x)|| = ||\phi(T_i x)||$ *for* $x \in \mathrm{X}$ *and* $i \in \{1, \ldots, n\}$. *Then* X *is already isometric to* H.

Proof. By putting $(x|y)_* = (\phi(x)|\phi(y))_{\mathrm{H}}$ for each $T_1, \ldots, T_n$ we obtain the assumptions of Theorem 2.2. Let $(\cdot|\cdot)_{\mathrm{X}} \colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ be the resulting inner product. Then X endowed with the norm $|||x||| \doteq \sqrt{(x|x)_{\mathrm{X}}}$ is transitive being a Hilbert space. Since X is maximally normed, we get that $\mathcal{G}_{(\mathrm{X},||\cdot||)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$. Thus X is transitive. It follows that $||\cdot|| = c|||\cdot|||$ for some $c > 0$, and hence X is a Hilbert space. □

Theorem 2.4. *Let* $(\mathrm{X}, ||\cdot||)$ *be a Banach space,* $(\mathrm{H}, (\cdot|\cdot)_{\mathrm{H}})$ *an inner product space,* $\mathcal{G} \subset \mathcal{G}(\mathrm{X})$ *a subgroup satisfying* $(*)$ *and let* $S \colon \mathrm{X} \to \mathrm{H}$ *be an isomorphism. Then there exists an inner product* $(\cdot|\cdot)_{\mathrm{X}}$ *on* X *such that*

(1) $||S^{-1}||^{-2}\ ||x||^2 \leq (x|x)_{\mathrm{X}} \leq ||S||^2\ ||x||^2$ *for* $x \in \mathrm{X}$.

(2) $(Tx|Ty)_{\mathrm{X}} = (x|y)_{\mathrm{X}}$ *for* $x, y \in \mathrm{X}$ *and* $T \in \overline{\mathcal{G}}^{\mathrm{SOT}} \subset L(\mathrm{X})$.

Proof. It suffices to find $(\cdot|\cdot)_{\mathrm{X}}$, which satisfies conclusion (1) and conclusion (2) for merely $T \in \mathcal{G}$. Indeed, given $T \in \overline{\mathcal{G}}^{\mathrm{SOT}}$ and $x, y \in \mathrm{X}$ there is a sequence $(T_n) \subset \mathcal{G}$ such that $T_n(x) \to T(x)$ and $T_n(y) \to T(y)$ as $n \to \infty$. This yields that $(T(x)|T(y))_{\mathrm{X}} - (x|y)_{\mathrm{X}} = \lim_{n\to\infty}((T_n(x)|T_n(y))_{\mathrm{X}} - (x|y)_{\mathrm{X}}) = 0$ by using the $\mathcal{G}$-invariance and the $||\cdot||$-continuity of $(\cdot|\cdot)_{\mathrm{X}}$.

Let $\mathcal{M}$ be the set of all pairs (E, G), where $E \subset \mathrm{X}$ is a finite-codimensional subspace and $G \subset \mathcal{G}$ is a finitely generated subgroup such that $T(E) = E$ for $T \in G$.

According to the definition of $\mathcal{G}$ we obtain that $\bigcup_{(E,G)\in\mathcal{M}} G = \mathcal{G}$ and $\bigcap_{(E,G)\in\mathcal{M}} E = \{0\}$. We equip $\mathcal{M}$ with the partial order $\leq$ defined as follows: $(E_1, G_1) \leq (E_2, G_2)$ if $E_1 \supset E_2$ and $G_1 \subset G_2$. So, $(\mathcal{M}, \leq)$ is a directed set.

Suppose that $\mathrm{Y} \subset \mathrm{H}$ is a subspace of a Hilbert space and H/Y is the corresponding quotient space. Then there exists a natural inner product on H/Y, namely

$$(\widehat{x}^{\mathrm{Y}}|\widehat{y}^{\mathrm{Y}})_{\mathrm{H}/\mathrm{Y}} = (x - P_{\mathrm{Y}}x|y - P_{\mathrm{Y}}y)_{\mathrm{H}}, \quad x, y \in \mathrm{H},$$

where $\widehat{x}^{Y} = x + \mathrm{Y}$, $\widehat{y}^{\mathrm{Y}} = y + \mathrm{Y}$ and $P_{\mathrm{Y}} \colon \mathrm{X} \to \mathrm{Y}$ is the orthogonal projection onto Y.

Given $(E, G) \in \mathcal{M}$ it holds that $T(E) = E$ for $T \in G$ and hence the mapping $\widehat{T}_E \colon \mathrm{X}/E \to \mathrm{X}/E$ given by $\widehat{T}_E(\widehat{x}^E) = T(x + E)$ defines a rotation on X/E for $T \in G$. Indeed, $||\widehat{x}^E||_{\mathrm{X}/E} = \mathrm{dist}(x, E)$ and $\mathrm{dist}(T(x), E) = \mathrm{dist}(x, E)$, as $T(E) = E$. Now, since X/E is finite-dimensional, the rotation group $\mathcal{G}_{\mathrm{X}/E}$ is compact in the operator norm topology.

For each $(E, G) \in \mathcal{M}$ we define a map $\widehat{S}_E \colon \mathrm{X}/E \to \mathrm{H}/S(E)$ by $\widehat{S}_E(\widehat{x}^E) = S(x + E)$. It is easy to see that

$$\begin{aligned} &||S^{-1}||^{-2}\ ||\widehat{x}^E||^2_{\mathrm{X}/E} \leq (\widehat{S}_E(\widehat{x}^E)|\widehat{S}_E(\widehat{x}^E))_{\mathrm{H}/S(E)} \\ &(\widehat{S}_E(\widehat{x}^E)|\widehat{S}_E(\widehat{y}^E))_{\mathrm{H}/S(E)} \leq ||S||^2\ ||\widehat{x}^E||_{\mathrm{X}/E}\ ||\widehat{y}^E||_{\mathrm{X}/E} \end{aligned} \tag{3}$$

for $x, y \in \mathrm{X}$. Consider $\mathbb{R}^{\mathcal{M}}$ with the point-wise linear structure. Define a map $B \colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}^{\mathcal{M}}$ by

$$B(x, y)(E, G) = \int_{\mathcal{G}_{\mathrm{X}/E}} (\widehat{S}_E(\tau\widehat{x}^E)|\widehat{S}_E(\tau\widehat{y}^E))_{\mathrm{H}/S(E)}\ \mathrm{d}\tau.$$

Above $\int_{\mathcal{G}_{X/E}}$ is the invariant Haar integral over the compact group $\mathcal{G}_{X/E}$. The invariance of the integral yields that $B(Tx,Ty)(E,G) = B(x,y)(E,G)$ for $x,y \in X$, $(E,G) \in \mathcal{M}$ and $T \in G$. By using (3) and the basic properties of the integral we obtain that

$$\begin{aligned} ||S^{-1}||^{-2}\ ||\widehat{x}^E||^2_{X/E} \leq B(x,x)(E,G) \\ B(x,y)(E,G) \leq ||S||^2\ ||\widehat{x}^E||_{X/E}\ ||\widehat{y}^E||_{X/E} \end{aligned} \tag{4}$$

for $x,y \in X$ and $(E,G) \in \mathcal{M}$.

The family $\{\{\gamma \in \mathcal{M}|\ \gamma \geq \eta\}\}_{\eta\in\mathcal{M}}$ is a filter base on $\mathcal{M}$. Let $\mathcal{U}$ be a non-principal ultrafilter extending $\{\{\gamma \in \mathcal{M}|\ \gamma \geq \eta\}\}_{\eta\in\mathcal{M}}$. Put $(x|y)_X = \lim_{\mathcal{U}} B(x,y)$ for $x,y \in X$. It is easy to see that $(\cdot|\cdot)_X$ is a bilinear mapping.

According to (4) we get that $(x|y)_X \leq ||S||^2||x||_X||y||_X$. Next, we aim to verify that $||S^{-1}||^{-2}||x||^2_X \leq (x|x)_X$. Towards this, we will check that $\sup_{(E,G)\in\mathcal{M}} ||\widehat{x}^E||_{X/E} = ||x||_X$. Fix $x \in \mathbf{S}_X$. Assume to the contrary that $\sup_{(E,G)\in\mathcal{M}} ||\widehat{x}^E||_{X/E} = c < 1$. Note that X is reflexive being isomorphic to H. Thus the ball $x + c\mathbf{B}_X$ is weakly compact. Putting

$$\{\{y \in E :\ ||x-y|| \leq C\}\}_{(E,G)\in\mathcal{M}}$$

defines a net of non-empty closed convex subsets of $x + c\mathbf{B}_X$. This net has a cluster point $z \in x + c\mathbf{B}_X$ according to the weak compactness of $x + c\mathbf{B}_X$. This means that $z \in \bigcap_{(E,G)\in\mathcal{M}} E$, which provides a contradiction, since $z \neq 0$. Consequently, (4) yields that

$$||S^{-1}||^{-2}\ ||x||^2_X = ||S^{-1}||^{-2}\ \lim_{\mathcal{U}} ||\widehat{x}^E||^2_{X/E} \leq \lim_{\mathcal{U}} B(x,x) = (x|x)_X.$$

Finally, we claim that $(Tx|Ty)_X = (x|y)_X$ for $x,y \in X$ and $T \in \mathcal{G}$. Indeed, pick $T \in \mathcal{G}$ and $x,y \in X$. Then

$$\begin{aligned} &\{(E,G) \in \mathcal{M} :\ B(T(x),T(y))(E,G) = B(x,y)(E,G)\} \\ \supset\ &\{(E,G) \in \mathcal{M} :\ T \in G\} \in \mathcal{U}, \end{aligned}$$

so that $\lim_{\mathcal{U}}(B(Tx,Ty) - B(x,y)) = 0$. □

Corollary 2.5. *Let* X *be a maximally normed space* X *isomorphic to a Hilbert space. Suppose that there is a subgroup* $\mathcal{G} \subset \mathcal{G}(X)$*, which satisfies* $(*)$ *and* $\mathcal{G}(X) \subset \overline{\mathcal{G}}^{SOT}$*. Then* X *is isometrically a Hilbert space.*

□

In Theorem 2.4 the isomorphism S was exploited in order to give bounds for the resulting inner product $(\cdot|\cdot)_X$. In [5] a different approach was taken instead; namely the analogous construction was suitably normalized by using a special point x_0. By suitably combining the arguments in [5] and in the proof of Theorem 2.4 we obtain the following result.

Theorem 2.6. *Let* X *be a Banach space transitive with respect to a subgroup* $\mathcal{G} \subset \mathcal{G}(X)$*, which satisfies* $(*)$*. Then* X *is isometric to a Hilbert space.*

□

Theorem 1.2 is an immediate consequence of the following result. This result yields that X must be in particular almost transitive, and we note that there exists an alternative route to this fact, since spaces both convex-transitive and superreflexive are additionally almost transitive, see e.g. [7].

Theorem 2.7. *Let* X *be a Banach space isomorphic to a Hilbert space and suppose* $\mathcal{G} \subset \mathcal{G}(\mathrm{X})$ *is a subgroup, which satisfies* $(*)$ *and* $\mathcal{G}_F \subset \mathcal{G}$. *Then* X *is convex-transitive with respect to* $\overline{\mathcal{G}}^{\mathrm{SOT}} \subset L(\mathrm{X})$ *if and only if* X *is isometric to a Hilbert space.*

Proof. First note that a Hilbert space is transitive, in particular convex-transitive, and that $\mathcal{G}_F \subset \mathcal{G}(\mathrm{H})$ is SOT-dense in $\mathcal{G}(\mathrm{H})$, so that the 'if' direction is clear.

Since X is isomorphic to a Hilbert space, we may apply Theorem 2.4 to obtain an $\overline{\mathcal{G}}^{\mathrm{SOT}}$-invariant inner product $(\cdot|\cdot)_{\mathrm{X}}$ on X such that $|||x|||^2 = (x|x)_{\mathrm{X}}$ defines a norm equivalent with $||\cdot||_{\mathrm{X}}$. Clearly $|||\cdot|||$ is $\overline{\mathcal{G}}^{\mathrm{SOT}}$-invariant as well. By rescaling $|||\cdot|||$ we may assume without loss of generality that $||\cdot||_{\mathrm{X}} \leq |||\cdot|||$ and $\sup_{y\in \mathbf{S}_{(\mathrm{X},|||\cdot|||)}} ||y||_{\mathrm{X}} = 1$. Put $C = \{x \in \mathrm{X} :\ |||x||| \leq 1\}$.

Fix $x \in \mathbf{S}_{(X,||\cdot||_{\mathrm{X}})}$ and $\epsilon > 0$. Let $y \in \mathbf{S}_{(\mathrm{X},|||\cdot|||)}$ be such that $||y||_{\mathrm{X}} > 1-\frac{\epsilon}{2}$. Since $(X, ||\cdot||_{\mathrm{X}})$ is convex-transitive with respect to $\overline{\mathcal{G}}^{\mathrm{SOT}}$, we get that $(1-\frac{\epsilon}{2})x \in \overline{\mathrm{conv}}^{||\cdot||_{\mathrm{X}}}(\{T(y)|T \in \overline{\mathcal{G}}^{\mathrm{SOT}}\})$. Since the norms $|||\cdot|||$ and $||\cdot||_{\mathrm{X}}$ are equivalent we obtain that there is a convex combination $\sum a_n T_n(y) \in \mathrm{conv}(\{T(y)|T \in \mathcal{G}_F\})$ such that $|||(1-\frac{\epsilon}{2})x - \sum a_n T_n(y)||| < \frac{\epsilon}{2}$. By noting that $|||\sum a_n T_n(y)||| \leq \sum a_n |||T_n(y)|||$ we get that $\sup_{T\in\overline{\mathcal{G}}^{\mathrm{SOT}}} |||T(y)||| \geq |||x||| - \epsilon$. Hence $|||y||| \geq |||x||| - \epsilon$ by using the $\overline{\mathcal{G}}^{\mathrm{SOT}}$-invariance of $|||\cdot|||$. Since ϵ was arbitrary and $|||x||| \geq 1$, we deduce that $|||x||| = 1$, and it follows that $||\cdot||_X = |||\cdot|||$. □

Finally, we will take a different approach and characterize the Hilbert spaces in terms of the subgroup of rotations, that, instead of fixing a finite-codimensional subspace, rather fix a given 1-dimensional subspace.

Proposition 2.8. *Let* X *be an almost transitive Banach space. Suppose that there exists* $z_0 \in \mathbf{S}_{\mathrm{X}}$ *satisfying that for any* $\epsilon > 0$ *and* $x, y \in \mathbf{S}_{\mathrm{X}}$ *with* $\mathrm{dist}(x, [z_0]) = \mathrm{dist}(y, [z_0]) = 1$, *there is* $T \in \mathcal{G}(\mathrm{X})$ *such that* $||T(z_0) - z_0|| < \epsilon$ *and* $||T(x) - y|| < \epsilon$. *Then* X *is isometric to an inner product space.*

Proof. It is well-known (see e.g. [3]) that almost transitive finite-dimensional spaces are isometric to Hilbert spaces. Hence we may concentrate on the case $\dim(\mathrm{X}) \geq 3$. Let $A, B \subset \mathrm{X}$ be 2-dimensional subspaces such that $z_0 \in A$. Recall the classical result that a Banach space is isometric to a Hilbert space if and only if any couple of 2-dimensional subspaces are mutually isometric (see [2]). Thus, in order to establish the claim, it suffices to verify that the subspaces A and B are isometric.

Fix $0 < \epsilon < 1$, $x \in \mathbf{S}_{\mathrm{X}} \cap A$ such that $\mathrm{dist}(x, [z_0]) = 1$ and $w \in \mathbf{S}_{\mathrm{X}} \cap B$. Let $f \in \mathbf{S}_{\mathrm{X}^*}$ be such that $f(w) = 1$.

Since X is almost transitive, there is $T_1 \in \mathcal{G}(\mathrm{X})$ such that $||T_1(w) - z_0|| < \frac{\epsilon}{4}$. Define a linear operator $S\colon \mathrm{X} \to \mathrm{X}$ by $S(v) = T_1(v) + f(v)(z_0 - T_1(w))$ for $v \in \mathrm{X}$ and note that $S(w) = z_0$. Observe that S is an isomorphism, since $||T_1 - S_1|| < \frac{\epsilon}{4}$. Pick $y \in \mathbf{S}_{\mathrm{X}} \cap S(B)$ such that $\mathrm{dist}(y, [z_0]) = 1$. According to the assumptions there is $T_2 \in \mathcal{G}(\mathrm{X})$ such that $\max(||T_2(z_0) - z_0||, ||T_2(y) - x||) < \frac{\epsilon}{4}$. Let $g, h \in 2\mathbf{B}_{\mathrm{X}^*}$ be such that $g(z_0) = h(y) = 1$, $y \in \mathrm{Ker}(g)$ and $z_0 \in \mathrm{Ker}(h)$. Define a linear operator $U\colon \mathrm{X} \to \mathrm{X}$ by

$$U(v) = T_2(v) + g(v)(z_0 - T_2(z_0)) + h(v)(x - T_2(y)) \quad \text{for } v \in \mathrm{X}.$$

Note that $U(z_0) = z_0$ and $U(y) = x$. Moreover, $||T_2 - U|| < \epsilon$, so that U is an isomorphism. Observe that $U \circ S$ maps B linearly onto A. We conclude that A and B are almost isometric, since ϵ was arbitrary. Hence, being finite-dimensional spaces, A and B are isometric. □

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