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WEAKLY COPRIME FACTORIZATION, CONTINUOUS-TIME SYSTEMS, AND STRONG-H^P AND NEVANLINNA FRACTIONS

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Abstract: *We give many necessary and sufficient conditions for the existence of a weakly coprime or of a Bézout coprime factorization of a transfer function, possibly operator-valued. Some of these conditions are given in terms of the output- or state-feedback stabilizability of realizations. Our realizations are well-posed linear systems—continuous-time linear time-invariant infinite-dimensional systems. We also study further properties of such factorizations, their relations to discrete-time weakly coprime factorizations, counter-examples and [weak] left-invertibility. Moreover, analogous discrete-time results are obtained. Control-theoretic consequences are indicated.*

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1 Introduction

In this article we study the properties of weakly and Bézout coprime factorizations and [weakly] left-invertible (possibly Hilbert space operator-valued) holomorphic functions and their relations to the LQ-optimal control and state-feedback stabilization of continuous-time systems. Some main results are new even for single-input-single-output systems.

Our class of realizations is the set of Well-Posed Linear Systems (or Abstract Linear Systems or Salamon–Weiss systems) [Sta05] [Sal87] [Wei94b] [Sta98a] [Mik06b] [WR00], [Sal89] [SW02] which are a generalization of the systems of the type $\dot{x} = Ax + Bu, \ y = Cx + Du$ and which allow the input and output operators B and C to be unbounded. They provide realizations for every function that is proper, i.e., holomorphic and bounded on a right half-plane. However, most of the article can be read without any knowledge on Well-Posed Linear Systems, and analogous results hold for many other classes too. Also analogous discrete-time results are provided.

In a "coprime factorization" $p = \frac{n}{m}$ of a rational number p, any common divisors of n and m (other than the units ± 1) have been canceled out, i.e., n and m are relative primes (coprime). Thus, their greatest common divisor is $\gcd(n, m) = 1$. Similarly, in a "right coprime factorization" $P = NM^{-1}$ of a function P, any common (right) divisors (in $\mathcal{H}^\infty$) of $N, M \in \mathcal{H}^\infty$ "have been canceled out". This means that

$$\text{if } \begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} V \text{ for some } A, B, V \in \mathcal{H}^\infty, \text{ then } V \text{ is a (right) divisor of } I, \tag{1}$$

i.e., then $I = LV$ for some $L \in \mathcal{H}^\infty$. This is sometimes expressed as "(a right) $\gcd(N, M) = I$". Condition (1) is equivalent to the *Bézout condition* that $XM - YN \equiv I$ for some $X, Y \in \mathcal{H}^\infty$ (take $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$, $V = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ to prove this).

If N and M are scalar-valued and we only require (1) to hold for scalar-valued functions V, then we get the classical definition of a "weakly right coprime factorization" [Fuh81] [Ino88] [Smi89]. The same holds in the matrix-valued case too if we only require (1) to hold for square-matrix-valued functions V, by Theorem 2.17(c).

In the operator-valued case one should only require (1) to hold when V^{-1} is proper (i.e., V has a uniformly bounded inverse on some right half-plane). That definition is equivalent to the classical one in the matrix-valued case (assuming that M^{-1} is proper), but it is "the right definition" in the operator-valued case too, in the sense that all functions of the form NM^{-1} do have weakly coprime factorizations and the classical relations to LQ-optimal state feedback are retained, etc. Most of these relations are new in the scalar-valued case too (for nonrational functions). However, we will use the property (2) below as our definition, because it is more useful in state-feedback contexts. We later prove our definition equivalent to the one above.

The theory on the connection between coprime factorization and different forms of stabilization of finite-dimensional systems became rather mature during the 70s and 80s [Vid85] [Fra87]. Thereafter, coprime factorization has played a major role in control theory, both finite- and infinite-dimensional. Also the infinite-dimensional setting has been studied intensively, but only now the theory is becoming complete.

The connection between dynamic stabilization and (Bézout) coprime factorization has been established also for general nonrational functions in, e.g., [Vid85], [Ino88], [Smi89], [Qua04] in the matrix-valued case, and in the operator-valued case in [CWW01] and [Mik07b]; all these for transfer functions only. Fairly general state-space results are given in [WR00].

In [CO06] and [Sta98a], certain connections between the coprime factorization and stabilizability and detectability were established. These results will be extended to an equivalence in Theorem 1.3.

In the finite-dimensional case, the coprime factorization of the transfer function of a system is determined by the LQ-optimal state feedback. In [Mik06b] the author showed that, in the infinite-dimensional case, the factorization defined by that state feedback is "weakly coprime" (not necessarily (Bézout) coprime, by Example 4.4).

Using this result, we establish algebraic and system-theoretic necessary and sufficient conditions for a (possibly operator-valued) function to have a (state-feedback) stabilizable realization or a weakly coprime factorization (Theorem 1.2). These results also provide important tools for the study of coprime factorizations and dynamic stabilization. We also study further properties of weakly coprime factorization and prove it equivalent to the classical "gcd = 1" definition in the matrix-value case [Fuh81] [Ino88] [Smi89]. Also similar results on Bézout coprime factorization are given.

Before presenting the main results, we need some definitions. Let $\mathtt{U}$, $\mathtt{X}$ and $\mathtt{Y}$ be arbitrary complex Hilbert spaces. Let $\mathcal{B}$ stand for bounded linear operators. Given $\omega \in \mathbb{R}$ we set $\mathbb{C}^+_\omega := \{z \in \mathbb{C} \bigm| \operatorname{Re} z > \omega\}$, and by $\mathcal{H}^\infty_\omega(\mathtt{U},\mathtt{Y})$ we denote the Banach space of bounded holomorphic functions $\mathbb{C}^+_\omega \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ with the supremum norm. We call P *proper* if $P \in \mathcal{H}^\infty_\infty := \cup_{\omega\in\mathbb{R}}\mathcal{H}^\infty_\omega$, i.e., if P is a bounded holomorphic function on some right half-plane. (We identify a holomorphic function on a right half-plane $\mathbb{C}^+_\omega$ with its restriction to any open subset of $\mathbb{C}^+_\omega$.)

The motivation to this is that the transfer functions of [stable] continuous-time systems are proper [and in $\mathcal{H}^\infty_0$]. Also the converse claims hold in our class of realizations.

A holomorphic function $f : \mathbb{C}^+_\omega \to \mathtt{U}$ is in $\mathcal{H}^2_\omega(\mathtt{U})$ if $\|f\|_{\mathcal{H}^2_\omega} := \sup_{r>\omega} \|f(r+i\cdot)\|_2 < \infty$. We set $\mathbb{C}^+ := \mathbb{C}^+_0$, $\mathcal{H}^2 := \mathcal{H}^2_0$, $\mathcal{H}^\infty := \mathcal{H}^\infty_0$, $\mathcal{H}^2_\infty := \cup_{\omega\in\mathbb{R}}\mathcal{H}^2_\omega$. By I we denote the identity operator $I \in \mathcal{B}$ or the constant function $I \in \mathcal{H}^\infty$.

We call NM^{-1} a (proper) *right factorization* (of P, if $P = NM^{-1}$ on a right half-plane) if $N \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$, $M \in \mathcal{H}^\infty(\mathtt{U})$ and M^{-1} is proper. If, in addition,

$$\begin{bmatrix} N \\ M \end{bmatrix} f \in \mathcal{H}^2 \implies f \in \mathcal{H}^2 \tag{2}$$

for each proper $\mathtt{U}$-valued f (or for each $f \in \mathcal{H}^2_\infty(\mathtt{U})$, by Theorem 3.4), then we call NM^{-1} a *w.r.c.f.* (weakly right coprime factorization).[1] An equivalent definition would be obtained with any $\mathcal{H}^p$ space in place of $\mathcal{H}^2$, by Theorem 9.8.

One can consider a w.r.c.f. of P as "the (algebraically) most canonical right factorization $\frac{N}{M}$" of P, i.e., as the one where any common nonregularities (e.g., zeros) of N and M "on $\overline{\mathbb{C}^+}$" have been canceled out. Several detailed interpretations of this are given later below.

In Section 2 (particularly in and below Theorems 2.14–2.17) we shall explore the different forms of coprimeness in detail, in particular, the above definition will be shown equivalent to the "$\gcd(N,M) = I$" definition presented further above (and to the classical one in the matrix-valued case). There we treat factorizations with $M(\alpha)$ invertible (for a fixed $\alpha \in \mathbb{C}^+$), but those with M^{-1} proper are treated in Theorem 3.2 and below Theorem 3.3.

The property (2) is very important from the control-theoretic point of view. In the literature that property (of Bézout coprime pairs) has been

[1] To be exact, we misuse the notation in a fairly standard way: we write $f \in \mathcal{H}^2(\mathtt{U})$ whenever $f : D(f) \to \mathtt{U}$ is such that $\mathbb{C}^+ \cap D(f)$ is nonempty and open and $f|_{\mathbb{C}^+\cap D(f)}$ is a restriction of an element of $\mathcal{H}^2$. That element of $\mathcal{H}^2$ is identified with f if these functions have a common holomorphic extension to some right half-plane, as mentioned above, but not in general, to avoid multi-valued functions (when f has different branches). Analogous misuse applies to $\mathcal{H}^\infty$ and to other spaces of holomorphic functions in place of $\mathcal{H}^2$.

used to reduce unstable control problems to stable ones. Now that can be done in the general case too, as explained in [Mik07g, Section 3].

A right factorization NM^{-1} is a *r.c.f.* if N and M are (Bézout) *right coprime (r.c.)*, i.e., if $\tilde{X}M - \tilde{Y}N = I$ for some $\tilde{X}, \tilde{Y} \in \mathcal{H}^\infty$. A r.c.f. is a w.r.c.f. (because then $f = \tilde{X}Mf - \tilde{Y}Nf \in \mathcal{H}^2$), but the converse requires additional assumptions (e.g., N, M being continuous on $\overline{\mathbb{C}^+} \cup \{\infty\}$ when $\dim \mathtt{U} < \infty$, see Theorem 2.7).

A right factorization NM^{-1} is *normalized* if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is *inner*, i.e., if $\| \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] u_0 \|_{\mathtt{Y}} = \|u_0\|_{\mathtt{U}}$ a.e. on the imaginary axis $i\mathbb{R}$ for every $u_0 \in \mathtt{U}$. (Here we used the fact that a Hilbert-space-valued $\mathcal{H}^\infty$ function has an L^∞ boundary function [RR85] [Mik08a].)

Now we can state our first main result: every right factorization can be made weakly coprime.

Theorem 1.1 *A $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function P has a right factorization iff it has a normalized weakly right coprime factorization.*

A normalized w.r.c.f. of P is unique modulo the right-multiplication by a unitary operator in $\mathcal{B}(\mathtt{U})$.

Moreover, if $P = NM^{-1}$ is a w.r.c.f., then all right factorizations of P are parameterized by $P = (NV)(MV)^{-1}$, where $V \in \mathcal{H}^\infty(\mathtt{U})$ and V^{-1} is proper. The w.r.c.f.'s are those for which $V^{-1} \in \mathcal{H}^\infty$ too. In particular, if a function P has a Bézout right coprime factorization, then every w.r.c.f. of P is Bézout right coprime.

Thus, $P = NM^{-1}$ is a w.r.c.f. iff M divides the denominator of every right factorization of P.

An intuitive control-theoretic implication is that a w.r.c.f. of P is the unique (modulo an invertible $V \in \mathcal{H}^\infty$) right factorization which "stabilizes" P (i.e., $N := PM \in \mathcal{H}^\infty$) with "as little effort as possible". Indeed, in the sense of the last paragraph of Theorem 1.1, the multiplier M is algebraically as close to the identity as possible, i.e., N is algebraically as close to P as possible.

From Theorem 1.1 we also conclude that if $P = NM^{-1}$ is a normalized w.r.c.f., then all normalized right factorizations of P are exactly those corresponding to an inner $V \in \mathcal{H}^\infty(\mathtt{U})$ such that V^{-1} is proper.

Now it is the time to explain some of the system- and control-theoretic properties and applications of w.r.c.f.'s. By Σ being a realization of P we mean that Σ is a well-posed linear system whose transfer function equals P. For brevity, we refer elsewhere some standard system-theoretic definitions.

The definitions of (iii)–(v) below and of *LQ-optimal* can be found in [Mik06b] or [Mik07d] or briefly in Section 5 (or, e.g., (iii)&(iv) in [Sta05], [Sta98a] or [Mik02] and (iii)&(v) in [CO06]).

Theorem 1.2 (w.r.c.f.) *The following are equivalent for any proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function P:*

(i) *P has a right factorization.*

(ii) *P has a normalized weakly right coprime factorization.*

(iii) *P has an output-stabilizable realization.*

(iv) *P has a stabilizable realization.*

(v) *P has a realization that satisfies the Finite Cost Condition.*

Moreover, if Σ is an output-stabilizable realization of P, then the LQ-optimal state feedback determines[2] *the normalized w.r.c.f. NM^{-1} of P, when we let N and M be the closed loop transfer functions from an extraneous input to the output and control, respectively, as described in [Mik06b] (or in Section 5).*

Conversely, every normalized w.r.c.f. corresponds to the LQ-optimal state-feedback for some system.

For example, the proper function $P(z) = \sqrt{z-1}$ does not satisfy (i)–(v), being not meromorphic on $\mathbb{C}^+$. However, without (i)–(v) most typical control problems on P do not have any solutions.

One more equivalent (to (i)) condition is that the (generalized) Hankel range of P is contained in the (generalized) Toeplitz range of P plus $\mathcal{H}^2$ (Theorem 6.3). We may also weaken (i) so that, instead of $N, M \in \mathcal{H}^\infty$, we only require that $f \in \mathcal{H}^2 \Rightarrow \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f(\omega + \cdot) \in \mathcal{H}^2 \ \forall \omega > 0$; a still "weaker" equivalent formulation is given in Theorem 6.2, and in Theorem 9.7 it will be shown, e.g., that it suffices to assume in (i) that N, M are strong-$\mathcal{H}^p$ or Nevanlinna functions to obtain (ii).

In the last result of this section, we shall present a similar equivalence on Bézout coprime factorizations. A map $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathtt{Y} \times \mathtt{U})$ such that $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty$ and M^{-1} is proper is called a *d.c.f.* (doubly coprime factorization) of NM^{-1}. It obviously follows that NM^{-1} is a r.c.f. (If we set $\left[\begin{smallmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{Y} & \tilde{X} \end{smallmatrix}\right] := \left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right]^{-1}$, then $\tilde{M}$ has a proper inverse and $\tilde{M}^{-1}\tilde{N}$ is called a l.c.f. of NM^{-1}.)

The terminology in (iii)–(v) below can be found in [Sta05], [Sta98a], [Mik02] or Section 5. [Input-]detectability is the dual of [output-]stabilizability.

Theorem 1.3 (r.c.f.) *The following are equivalent for a (proper) function P:*

(i) *P has a r.c.f. (right coprime factorization).*

(ii) *P has a d.c.f. (doubly coprime factorization).*

(iii) *P has an output-stabilizable and input-detectable realization.*

---

[2]The LQ-optimal state-feedback is unique modulo an invertible constant [Mik06b, Lemma A.5]. Therefore, depending on the choice of the LQ-optimal feedback, we can actually obtain the maps NE and ME for any invertible $E \in \mathcal{GB}(\mathtt{U})$. If we have no feedthrough in the feedback loop and, e.g., the input operator is bounded, then $M^*M + N^*N = I + P(+\infty)^*P(+\infty)$ "a.e." on the imaginary axis. Consequently, to have NM^{-1} normalized we must have a feedthrough in the feedback loop unless the original feedthrough $P(+\infty)$ is zero. Further details are given in [Mik06b], where also "(v)⇒(ii)" was established.

(iv) *P has a stabilizable and detectable realization.*

(v) *P has a jointly stabilizable and detectable realization.*

Thus, a r.c.f. means the w.r.c.f. of a function having a stabilizable and detectable realization. (It is not sufficient to have a stabilizable and a (different) detectable realization [Mik07d].)

A sixth equivalent condition is that P is dynamically stabilizable (i.e., that $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty$ for some Q), as will be shown in [Mik07b] using Theorems 1.1 and 1.3. For matrix-valued functions the sufficiency of dynamic stabilization was established in [Ino88] and [Smi89], the necessity and more in [Tre92] (whose extension to the matrix-valued case is contained in [Vas71], as noted in [Qua04]).

As explained here and in [Mik07g], in most algebraic and system-theoretic aspects, weak coprimeness is the more canonical generalization of finite-dimensional coprimeness. However, when it comes to dynamic stabilization, Bézout coprimeness is a necessary (and in most cases also sufficient) requirement. Yet w.r.c.f.'s are an important tool for proving such connections.

Notes

Naturally, to every "right" definition (e.g., "r.c.f.") or result in this article there exists a corresponding "left" definition or result, by duality (replace P (resp., $M, N, \dots$) by P^{d} (resp., $M^{\mathrm{d}}, N^{\mathrm{d}}, \dots$), where $P^{\mathrm{d}}(z) := P(\bar{z})^*$). In particular, the existence of a "l.c.f." is one more equivalent condition in Theorem 1.3.

We call here $N, M \in \mathcal{H}^\infty$ *gcd-w.r.c.* if (1) holds for every square-matrix-valued V. For gcd-w.r.c.f.'s, the first result of Theorem 1.1 was established in [vR77] in the scalar-valued case and in [Ino88] and [Smi89] in the matrix-valued case, independently. In Theorem 3.1(c) we prove the equivalence of "gcd-w.r.c." and (our definition of) "w.r.c." (using Theorem 1.1) assuming that $\dim \mathtt{U} < \infty$. When $\dim \mathtt{U} = \infty$, "gcd-w.r.c." is equivalent to "r.c." instead, by Theorem 2.17(b). The discrete-time forms of Theorems 1.1–1.3 and many other our results are given already in [Mik07g], which also contains further historical comments on them.

The widespread implicit use of the property (2) (of r.c. pairs) in control-theoretic literature, to reduce unstable problems to stable ones, was the reason for its explicit use in [Mik02] as a separate definition.

Theorem 1.2 is otherwise new but it can be derived from the author's earlier works [Mik06b] [Mik02].

In the matrix-valued case, the implication "(i)⇔(ii)" in Theorem 1.3 is a direct consequence of Tolokonnikov's Lemma [Tol81]. The lemma was extended to operator-valued functions in [Tre04] (the nonseparable case in [Mik09]), from which we derive the equivalence (Lemma 5.2). The implication "(iii)⇒(ii)" was established in [CO06] (assuming that A is invertible, but our proof is based on their ideas). The equivalence "(ii)⇔(v)" is [Sta98a, Theorem 4.4]. The remaining implications are trivial.

Although this introduction is superficially similar to that of [Mik07g], our continuous-time setting is much more difficult than the discrete-time setting of [Mik07g] due to unbounded operators in the realizations. However, the smaller classes (than the class of well-posed linear systems) used in the literature could not realize all proper transfer functions (hence nor make Theorems 1.2 and 1.3 true). Moreover, the remaining sections contain many results that do not even have a counterpart in [Mik07g]. Most of them provide new discrete-time results too, as noted in Remark 7.6. Nevertheless, further details or explanations on some "common" results are given in [Mik07g].

Most of our results are new even in the scalar-valued case. However, it has become common in system-theoretic or PDE-oriented systems and control theory to have infinite-dimensional input and output spaces [Fuh81] [CZ95] [LT00] [Sta05]. They can be used to describe and study natural phenomena, including feedback.

For example, one often has interconnected systems, such as an acoustic cavity with one wall being a flexible membrane (or a plate). To describe this system, one regards the wave equation in the cavity as one system and the 2-dimensional wave (or plate) equation for the wall as another. These two systems interact via infinite-dimensional signals (the pressure distribution along the wall, and the velocity distribution along the wall). There are many references for such coupled systems, which are called "structural acoustic systems"; see, e.g., [ALR03] and its references.

Another example is that one may create a static boundary feedback physically. For example, one may put an energy absorbing coating on the walls of a cavity with waves inside. For electromagnetic waves, this would be a resistive coating, for acoustic waves, one would put a sound absorbing material. This can be described mathematically as a static feedback (from output to input) where the input and output spaces are infinite-dimensional. Such feedbacks have been studied by Ammari and Tucsnak, Komornik, Guo and Zhang, Luo and others.

In Section 2 we establish several properties of α-[w.]r.c.f.'s, where M is required to be invertible at a fixed $\alpha \in \mathbb{C}^+$ only (and $Nf, Mf \in \mathcal{H}^2 \Rightarrow f \in \mathcal{H}^2$ needs to hold for functions f holomorphic on a neighborhood of α). These correspond to transfer functions of possibly ill-posed systems but also provide a bridge between continuous- and discrete-time results. Many results are established for α-weak left-invertibility, which is a generalization of α-weak coprimeness.

Section 3 contains corresponding results for (proper) w.r.c.f.'s (and for weak left-invertibility). Also the relations between these two concepts are treated and Theorem 1.1 is proved.

In Section 4 we present counter-examples illustrating that many of our results are optimal. For example, we note that w.r.c.f.'s need not be r.c.f.'s and α-w.r.c.f.'s need not be w.r.c.f.'s. We also construct a SISO system whose LQ-optimal feedback determines a noncoprime w.r.c.f.

In Section 5 we present well-posed linear systems and prove Theorems 1.2 and 1.3 and related results.

Section 6 contains further conditions equivalent to Theorem 1.2(i) and related results. Moreover, constructive algorithms are given for a w.r.c.f. and for an output-stabilizable realization (resp., for a r.c.f., a d.c.f., a robust stabilizing controller and a stabilizable and detectable realization) of a given function satisfying the assumptions of Theorem 1.2 (resp., of Theorem 1.3).

In Section 7 we present further results on left invertibility and coprime factorization, both standard, weak and α-weak: the properties of their reciprocals and Hankel operators, the relations between different values of α, and conditions under which lack of zeros implies left-invertibility in the operator-valued case. These results (and some others) are equally applicable in the discrete-time setting.

In Section 9 we present still further conditions equivalent to Theorem 1.2(i), namely that $P = NM^{-1}$, where N and M lie in the Nevanlinna class or in strong-$\mathcal{H}^p$ for some $p \geq 2$. Also Theorem 1.1 and some other main results are generalized to these classes in place of $\mathcal{H}^\infty$. Furthermore, we show that, in (2) (i.e., in the definition of weak coprimeness), one can replace $\mathcal{H}^2$ by any $\mathcal{H}^p$ or $\mathcal{H}^p_{\text{strong}}$ ($1 \leq p \leq \infty$). Analogous discrete-time results are given in Section 8.

The contents of this report will be published as [Mik08b] with same theorem numbers except that Lemma 7.8 and Sections 8 and 9 will be omitted.

Notation. We present the following terminology in the following order. Section 1: $\mathcal{B}$, $\mathtt{U}$, $\mathtt{X}$, $\mathtt{Y}$, $\mathbb{C}^+_\omega$, proper, $\mathcal{H}^\infty_\omega$, $\mathcal{H}^\infty_\infty$, $\mathcal{H}^2_\omega$, $\mathbb{C}^+$, $\mathcal{H}^2$, $\mathcal{H}^\infty$, $\mathcal{H}^2_\infty$, I, right factorization, w.r.c.f., (Bézout) r.c.f., normalized, inner, (LQ-optimal), d.c.f. Section 2: α-right factorization, $\mathcal{GB}$, α-proper, α-weakly left-invertible, coercive, left-invertible, α-w.r.c., α-w.r.c.f., α-r.c.f., $\mathcal{G}$, $\mathcal{G}\mathcal{H}^\infty$, outer, irreducible, gcd-coprime right divisor, square (right divisor), divisor-left-invertible, (dual) inner-outer factorization. Section 3: α-d.c.f., weakly left-invertible, w.r.c. Section 5: L^2_ω, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$, $\widehat{\ } : u \mapsto \widehat{u}$, $\mathscr{D}_P$, τ^t, π_+, π_-, WPLS, Σ, $\left[\begin{smallmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\cdot & \cdot\\ \cdot & \cdot\end{smallmatrix}\right]$, dual Σ^{d}, A, B, C, D, $\hat{\mathscr{D}}$, $\overline{C}$, transfer function, stable, state feedback, output-stable, [output]-stabilizable, detectable, Finite Cost Condition, LQ-optimal, realization. Section 6: $\mathrm{L}^2_{\mathrm{c}}$, $\mathbb{D}$. Section 8: $\mathcal{H}^p$, $\mathtt{B}$, discrete-time form, Nev, Nev_+, $\mathrm{Nev}_{\text{strong}}$, $\mathrm{Nev}_{+,\text{strong}}$, $\mathcal{H}^p_{\text{strong}}$. Section 9: (continuous-time variants of) $\mathcal{H}^p$, Nev, Nev_+, $\mathrm{Nev}_{\text{strong}}$, $\mathrm{Nev}_{+,\text{strong}}$, $\mathcal{H}^p_{\text{strong}}$.

2 α-weak left-invertibility and α-w.r.c.

The transfer functions of many ill-posed systems are not proper, hence they do not have right factorizations. Nevertheless, they are usually holomorphic functions $P : \Omega \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ on a neighborhood Ω of some $\alpha \in \mathbb{C}^+$ and they often have *α-right factorizations* NM^{-1}, which means that $N \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$, $M \in \mathcal{H}^\infty(\mathtt{U})$, $M(\alpha) \in \mathcal{GB}(\mathtt{U})$ and $P = NM^{-1}$. Here $\mathcal{G}$ stands for (the $\mathcal{G}$roup of) invertible elements.

In this section we present certain properties of such factorizations. We prove only some of the results explicitly and mention some others as their direct corollaries. The remaining results are essentially the same as analogous discrete-time results in [Mik07g] (in particular, the same proofs apply, mutatis mutandis; alternatively, Lemma 9.1 can be used). By Theorem 3.2 (and 3.1(b)), the results given for α-right factorizations in this section do hold for (proper) right factorizations too; also the "proper" forms of the other results are treated in Section 3.

Throughout this section we assume that $\alpha \in \mathbb{C}^+$. A holomorphic function defined on a neighborhood of α is called *α-proper.* We call $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ *α-weakly left-invertible* if $Ff \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$ for every α-proper $\mathtt{U}$-valued f, and $F(\alpha)$ is *coercive* (i.e., $F(\alpha)^*F(\alpha) \geq \epsilon I$ for some $\epsilon > 0$). The function F is *left-invertible* if $GF = I$ for some $G \in \mathcal{H}^\infty(\mathtt{Y},\mathtt{U})$, *invertible* if $F \in \mathcal{G}\mathcal{H}^\infty$ (i.e., $GF = I = FG$ for some $G \in \mathcal{H}^\infty$).

We call $N, M \in \mathcal{H}^\infty$ *α-w.r.c.* if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is α-weakly left-invertible. An α-right factorization NM^{-1} is called an *α-w.r.c.f.* (resp., *α-r.c.f.*) iff $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is α-weakly left-invertible (resp., left-invertible). (Note that an α-right factorization is an α-w.r.c.f. iff $Nf, Mf \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$ for every proper $\mathtt{U}$-valued f, because an invertible $M(\alpha)$ is coercive.) Thus, all our results for α-weak left-invertibility trivially lead to analogous corollaries on α-weak coprimeness, although those of Corollaries 2.4 and 2.5, Theorem 2.10 and Lemma 2.11 are hardly interesting.

Theorem 1.1 also holds for α-w.r.c.f.'s:

Theorem 2.1 *A $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function P has an α-right factorization iff it has a normalized α-w.r.c.f.*

A normalized α-w.r.c.f. of P is unique modulo the right-multiplication by a unitary operator in $\mathcal{B}(\mathtt{U})$.

Moreover, if $P = NM^{-1}$ is an α-w.r.c.f., then all α-right factorizations of P are parameterized by $P = (NV)(MV)^{-1}$, where $V \in \mathcal{H}^\infty(\mathtt{U})$ and V^{-1} is α-proper. The α-w.r.c.f.'s are those for which $V^{-1} \in \mathcal{H}^\infty$ too. In particular, if a function P has an α-r.c.f., then every α-w.r.c.f. of P is an α-r.c.f.

An α-weakly left-invertible function is one-to-one on $\mathbb{C}^+$ and coercive on the boundary:

Theorem 2.2 (No zeros) *If $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is α-weakly left-invertible, then there exists $\epsilon > 0$ such that, for every $u_0 \in \mathtt{U} \setminus \{0\}$, we have $\|Fu_0\| \geq \epsilon \|u_0\|$ a.e. on $i\mathbb{R}$ and $F(z)u_0 \neq 0$ for every $z \in \mathbb{C}^+$.*

Thus, α-w.r.c. functions do not have "common zeros" on $\mathbb{C}^+$. The converse is not true; e.g., $F(z) = \mathrm{e}^{-z}$ is coercive on $\mathbb{C}^+$ and inner but not α-weakly left-invertible, by Example 4.1. A Tauberian converse is given in Theorem 2.7.

Lemma 2.3 *If $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is α-weakly left-invertible and R is α-proper and $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued, then $FR \in \mathcal{H}^\infty \Leftrightarrow R \in \mathcal{H}^\infty$. Moreover, then FR is α-weakly left-invertible iff R is α-weakly left-invertible.*

(A converse is given in Theorem 2.14.)

Recall that $\mathcal{GV}(\mathtt{U},\mathtt{Y}) := \{F \in \mathcal{V}(\mathtt{U},\mathtt{Y}) \,\big|\, GF = I \;\&\; FG = I$ for some $G \in \mathcal{V}(\mathtt{Y},\mathtt{U})\}$ when $\mathcal{V} = \mathcal{B}$ or $\mathcal{V} = \mathcal{H}^\infty$. (Left-)invertibility in $\mathcal{H}^\infty$ obviously implies α-weak left-invertibility. We get the converse by assuming that F is invertible at α:

Corollary 2.4 *If $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is α-weakly left-invertible and $F(\alpha) \in \mathcal{GB}(\mathtt{U}, \mathtt{Y})$, then $F \in \mathcal{G}\mathcal{H}^\infty$.*

If $F(\alpha)$ is a square matrix, the second assumption becomes redundant, by Theorem 2.2:

Corollary 2.5 *If $F \in \mathcal{H}^\infty(\mathbb{C}^n)$ is α-weakly left-invertible, then $F \in \mathcal{G}\mathcal{H}^\infty$.*

However, α-weak left-invertibility does not imply left-invertibility for non-square functions, by Example 4.2, nor for elements of $\mathcal{H}^\infty(\mathtt{U})$ with $\dim \mathtt{U} = \infty$, by Example 4.7.

The "Corona condition" lies between α-weak and usual left-invertibility:

Lemma 2.6 (Corona) *Let $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$.*

(a) *If $GF = I$ for some $G \in \mathcal{H}^\infty(\mathtt{Y}, \mathtt{U})$, then there exists $\epsilon > 0$ such that $F^*F \geq \epsilon I$ on $\mathbb{C}^+$. The converse holds if $\dim \mathtt{U} < \infty$.*

(b) *If $F^*F \geq \epsilon I$ on $\mathbb{C}^+$, then F is α-weakly left-invertible.*

If $\dim \mathtt{U} = \infty$, then the converse in Lemma 2.6(a) is no longer true [Tre89]. The converse to (b) is not true at all, by (a) and Example 4.2.

For functions in the (matrix-valued) "half-plane algebra", hence for all rational functions, α-weak left-invertibility is equivalent to left-invertibility as well as to F having no zeros on $\overline{\mathbb{C}^+} \cup \{\infty\}$:

Theorem 2.7 ("Disc algebra") *Assume that $\dim \mathtt{U} < \infty$ and $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$. If F is continuous on $K := \overline{\mathbb{C}^+} \cup \{\infty\}$, or F is continuous on some (other) closed $K \subset \overline{\mathbb{C}^+} \cup \{\infty\}$ and there exists $\epsilon > 0$ such that $F^*F \geq \epsilon I$ on $\mathbb{C}^+ \setminus K$, then the following are equivalent:*

(i) *$GF = I$ for some $G \in \mathcal{H}^\infty$.*

(ii) *For any open $\Omega \subset \mathbb{C}^+$ and any function $f : \Omega \to \mathtt{U}$ we have $Ff \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$.*

(iii) *F is α-weakly left-invertible.*

(iv) *$F(z)u_0 \neq 0$ for all $z \in K$ and all $u_0 \in \mathtt{U} \setminus \{0\}$.*

(Note in (ii) that "$\in \mathcal{H}^2$" means being a restriction of an $\mathcal{H}^2$ function.)

Because of this fact, the difference between a r.c.f. and an α-w.r.c.f. becomes redundant in the finite-dimensional systems and control theory (cf. [Fra87]).

An α-w.r.c.f. is a z-w.r.c.f. for any reasonable $z \in \mathbb{C}^+$:

Theorem 2.8 *Let $\alpha \in \mathbb{C}^+$, $M \in \mathcal{H}^\infty(\mathtt{U})$. Then the following hold:*

(a) *Assume that NM^{-1} is an α-right factorization, $N_0M_0^{-1}$ is an α-w.r.c.f. of NM^{-1}, $\Omega \subset \mathbb{C}^+$ is open and connected and $\alpha \in \Omega$. If M is invertible on Ω, then so is M_0; if M^{-1} is uniformly bounded on Ω, then so is M_0^{-1}.*

If $\dim \mathtt{U} < \infty$, then Ω need not be connected above.

(b) *If NM^{-1} is an α-w.r.c.f. and M is invertible on an open and connected $\Omega \subset \mathbb{C}^+$ such that $\alpha \in \Omega$, then NM^{-1} is a z-w.r.c.f. for every $z \in \Omega$.*

If $\dim \mathtt{U} < \infty$, then Ω need not be connected above and, in addition, N and M are z-w.r.c. for every $z \in \mathbb{C}^+$.

From Theorem 2.2 it follows that the outer factor of F is invertible (see [Sta97, Lemma 18(ii)] or [Mik09, below Theorem 5.11]). Therefore, F can be normalized as follows:

Lemma 2.9 (Inner) *If $\alpha \in \mathbb{C}^+$ and $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is α-weakly left-invertible, then there exists $S \in \mathcal{G}\mathcal{H}^\infty(\mathtt{U})$ such that FS is α-weakly left-invertible and inner.*

A function $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is called *outer* iff $\{Ff \mid f \in \mathcal{H}^2(\mathtt{U})\}$ is dense in $\mathcal{H}^2(\mathtt{Y})$.

Any $F \in \mathcal{H}^\infty$ that is bounded below at some point $\alpha \in \mathbb{C}^+$ can be factorized as follows:

Theorem 2.10 (F=F$_w$F$_r$) **(a)** *If $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$, and $F(\alpha)$ is coercive for some $\alpha \in \mathbb{C}^+$, then $F = F_w F_r$, where $F_w \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is inner and α-weakly left-invertible, $F_r \in \mathcal{H}^\infty(\mathtt{U})$ and $F_r(\alpha) \in \mathcal{GB}(\mathtt{U})$.*

All such factorizations are given by $F = (F_w V)(V^{-1} F_r)$, where $V \in \mathcal{B}(\mathtt{U})$ is unitary (or $V \in \mathcal{G}\mathcal{H}^\infty(\mathtt{U})$ if we do not require F_w to be inner). In particular, if some left factor $F_w V$ is left-invertible in $\mathcal{H}^\infty$, then so is every $F_w V$.

(b) *If $F \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$, then there exist $\alpha \in \mathbb{C}^+$ and $m \le n$ such that $F = F_w F_r F_o$, where $F_o \in \mathcal{H}^\infty(\mathbb{C}^n, \mathbb{C}^m)$ is outer, $F_w \in \mathcal{H}^\infty(\mathbb{C}^m, \mathtt{Y})$ is inner and z-weakly left-invertible for every $z \in \mathbb{C}^+$, $F_r \in \mathcal{H}^\infty(\mathbb{C}^m)$ is inner, and $F_r(\alpha) \in \mathcal{GB}(\mathbb{C}^m)$.*

All such factorizations are given by $F = (F_w V)(V^{-1} F_r W^{-1})(W F_o)$, where $V, W \in \mathcal{B}(\mathbb{C}^m)$ are unitary.

(c) *In (a) we have $F_w = J \left[\begin{smallmatrix} N \\ M \end{smallmatrix} \right]$, where $J \in \mathcal{B}(\mathtt{Y}_1 \times \mathtt{U}, \mathtt{Y})$ is unitary, $\mathtt{Y}_1 \subset \mathtt{Y}$ is a closed subspace, $\mathtt{Y}_1^\perp$ is isometric to $\mathtt{U}$ and NM^{-1} is a $\mathcal{B}(\mathtt{U}, \mathtt{Y}_1)$-valued α-w.r.c.f.*

Next we extend Theorem 2.10(a) to apply on a set instead of a single point:

Lemma 2.11 *If $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is coercive on an open, connected set $\Omega \subset \mathbb{C}^+$, then $F = F_w F_r$, where $F_r \in \mathcal{H}^\infty(\mathtt{U})$, $F_w \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is inner and z-weakly left-invertible and $F_r(z) \in \mathcal{GB}(\mathtt{U})$ for every $z \in \Omega$. If $F^* F \ge \epsilon^2 I$ on Ω, then $\|F_r^{-1}\| \le \epsilon^{-1}$ on Ω.*

Proof of Lemma 2.11: Choose some $\alpha \in \Omega$ and write $F = F_w F_r$ as in Theorem 2.10(a). The function $G := (F^* F)^{-1} F^* F_w : \Omega \to \mathcal{B}(\mathtt{U})$ is continuous on Ω and $G F_r = I$ on Ω, hence $F_r G = I$ at α (because $F_r(\alpha) \in \mathcal{GB}(\mathtt{U})$). It obviously follows that $\Omega_r := \{z \in \Omega \mid F_r(z) \in \mathcal{GB}(\mathtt{U})\}$ must equal Ω (because $F_r G = I$ on $\partial\Omega_r \cap \Omega$ and $\Omega_r \ni \alpha$ is open). Since F_w is inner, we have $\|F_r(s)^{-1}\| \le \epsilon^{-1}$ if $F(s)^* F(s) \ge \epsilon^2 I$.

Let $z \in \Omega$ be arbitrary and get $F = F_w' F_r'$ from Theorem 2.10(a). Now $V := F_r' F_r^{-1}$ z-proper and $F_w = F_w' V$, hence $V \in \mathcal{H}^\infty(\mathtt{U})$, by Lemma 2.3. Similarly, $V^{-1} \in \mathcal{H}^\infty(\mathtt{U})$, hence F_w is z-weakly left-invertible. □

If, in addition, F is α-weakly invertible for some $\alpha \in \Omega$, then $F_r \in \mathcal{G}\mathcal{H}^\infty$, by Theorem 2.1. This proves the following:

Corollary 2.12 (Every α) *Assume that $\Omega \subset \mathbb{C}^+$ is open and connected, $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$, and F is coercive on Ω. If F is α-weakly left-invertible for some $\alpha \in \Omega$, then F is z-weakly left-invertible for every $z \in \mathbb{C}^+$.*

The above coercivity assumption is not superfluous in general, by Example 4.6, but it is redundant if $\dim \mathtt{U} < \infty$, by Theorem 2.2, hence we have the following:

Corollary 2.13 (Every α) *If $F \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$ is α-weakly left-invertible for some $\alpha \in \mathbb{C}^+$, then F is z-weakly left-invertible for every $z \in \mathbb{C}^+$.*

Thus, for matrix-valued functions the point $\alpha \in \mathbb{C}^+$ does not have a special role. The function F does not have to be uniformly coercive in either corollary, by Example 4.2 (cf. Remark 3.6). In Corollary 2.13, a third equivalent condition is Theorem 2.7(ii) (or the same condition with $\mathcal{H}^\infty$ in place of $\mathcal{H}^2$).

In the definition of α-weak left-invertibility (or α-w.r.c.f.'s) we could have $\mathcal{H}^\infty$ in place of $\mathcal{H}^2$:

Theorem 2.14 *Assume that $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$, $F(\alpha)$ is coercive and $\mathtt{X} \neq \{0\}$. Then F is α-weakly left-invertible iff $FR \in \mathcal{H}^\infty \Rightarrow R \in \mathcal{H}^\infty$ for every α-proper $\mathcal{B}(\mathtt{X}, \mathtt{U})$-valued function R.*

The coercivity assumption is necessary, because $F(z) = z - \alpha$ is not α-weakly left-invertible.

Now we present the classical definition of weak coprimeness (of matrix-valued functions) [Fuh81] [Ino88] [Smi89] and show that it is equivalent to ours. We call $F \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$ *irreducible* if $Ff \in \mathcal{H}^\infty \implies f \in \mathcal{H}^\infty$ holds for every function of the form $f = g^{-1}G$, where $0 \not\equiv g \in \mathcal{H}^\infty(\mathbb{C})$, $G \in \mathcal{H}^\infty(\mathbb{C}, \mathbb{C}^n)$. If $\dim \mathtt{Y} < \infty$, then F is irreducible iff 1 is a gcd of all highest order minors of F [Smi89, Lemma 4]. We call functions $N \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$ and $M \in \mathcal{H}^\infty(\mathbb{C}^n)$ *gcd-coprime* iff $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is irreducible. This is the classical definition of (right) "weak coprimeness".

The following theorem shows that the factorizations of [Smi89, Lemma 5] is a special case of Theorem 2.10(a). Therefore, the "weakly coprime right factorization" of [Smi89, p. 1007] is the same as an α-w.r.c.f. (when $\dim \mathtt{U}, \dim \mathtt{Y} < \infty$, as [Smi89] assumes).

Theorem 2.15 (gcd-coprime) *Let $F \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$. Then F is irreducible iff for some (hence every) $\alpha \in \mathbb{C}^+$ the function F is α-weakly left-invertible.*

In particular, functions $M \in \mathcal{H}^\infty(\mathbb{C}^n)$ and $N \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$ are gcd-coprime iff they are α-w.r.c. for some (hence every) $\alpha \in \mathbb{C}^+$.

There are also many other natural definitions of (right) coprimeness than r.c., w.r.c. and gcd-coprimeness. One of them is the "$\gcd(N, M) = I$" condition (1) in the introduction. The above and the next two theorems present the relations between such competing definitions.

If $F = LR$, where $F, L, R \in \mathcal{H}^\infty$, then we call R a *right divisor* of F; it is *square* if $R \in \mathcal{H}^\infty(\mathtt{X})$ for some Hilbert space $\mathtt{X}$ (i.e., if its input and output spaces are the same). It inherits α-weak left-invertibility from F (if any):

Theorem 2.16 (Divisors) *A function $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is α-weakly left-invertible iff every square right divisor of F is α-weakly left-invertible, or equivalently, iff every right divisor of F is α-weakly left-invertible.*

A function $F \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$ is α-weakly left-invertible iff every square right divisor of F is invertible.

Therefore, an α-right factorization NM^{-1} is a w.r.c.f. iff every square right divisor of $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is α-weakly left-invertible (or invertible, if $\dim \mathtt{U} < \infty$). Thus, the word "coprime" in "w.r.c." is justified in a weak sense. Note that in the scalar-valued case the word "right" is usually removed and that in the operator-valued case "weakly left" is not redundant (some right divisors are not right-invertible).

Indeed, a function $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is α-weakly left-invertible iff all "α-invertible" right divisors of F are invertible (i.e., $F = LR$, $F, L, R \in \mathcal{H}^\infty$, $R(\alpha) \in \mathcal{GB} \Rightarrow R \in \mathcal{G}\mathcal{H}^\infty$), by Theorem 2.16 and Corollary 2.4. Without the invertibility assumption on $R(\alpha)$ we could have, e.g., $F = I$, $R =$right shift, $L =$left shift on $\mathtt{U} = \ell^2(\mathbb{N})$. Thus, when $\dim \mathtt{U} = \infty$, some square right divisors of any F are non-invertible, but the more natural interpretation of gcd-coprimeness becomes equivalent to coprimeness, as observed below Theorem 2.17.

Proof of Theorem 2.16: 1° *Right divisors:* Let $F = LR$. If $Rf \in \mathcal{H}^2$, then $Ff = LRf \in \mathcal{H}^2$, so if F is α-weakly left-invertible, then so is R (because $R(\alpha)$ is coercive if $F(\alpha)$ is coercive). The converse is obvious, because $F = IF$.

2° *Case $F \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$:* "Only if" follows from 1° and Corollary 2.5, so assume that every square right divisor of F is invertible. From Theorem 2.10(b) we obtain a factorization $F = F_w F_r F_0$, i.e., $F = \begin{bmatrix} F_w & 0 \end{bmatrix} \left[\begin{smallmatrix} F_r & 0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} F_o \\ 0 \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} F_o & 0 \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathbb{C}^n)$ is square, hence invertible (in $\mathcal{H}^\infty$). Therefore, $m = n$, so $F_r F_o$ is square, hence invertible, so $F = F_w(F_r F_o)$ is z-weakly left-invertible (because so is F_w) for every $z \in \mathbb{C}^+$.

3° *Square right divisors:* By the above, it remains to assume that $\dim \mathtt{U} = \infty$ and that every square right divisor of F is α-weakly left-invertible, and to show that F is α-weakly left-invertible.

Since $\mathbb{C}^+$ is separable and $\dim \mathtt{U}$ is infinite, the closed span $\mathtt{Y}_0$ of $\cup_{s \in \mathbb{C}^+} F(s)$ has $\dim \mathtt{Y}_0 \le \dim \mathtt{U}$. Consequently, there exists $T \in \mathcal{B}(\mathtt{Y}_0, \mathtt{U})$ such that $T^*T = I$ (this is elementary [Mik02, Lemma A.3.1(a4)]). Let $P \in \mathcal{B}(\mathtt{Y}, \mathtt{Y}_0)$ be the orthogonal projection $\mathtt{Y} \to \mathtt{Y}_0$. Since $F = P^*T^*TPF$, and $TPF \in \mathcal{H}^\infty(\mathtt{U})$ is square, TPF is α-weakly left-invertible. Therefore, $TPFf \notin \mathcal{H}^2$ for each α-proper $f \notin \mathcal{H}^2$, hence $Ff = P^*T^*TPFf \notin \mathcal{H}^2$ for such f (for $g := PFf$

we have $\|Ff\|_2 = \|Tg\|_2 = \infty$, because $\|T^*Tg\|_2 = \|g\|_2 = \|Tg\|_2$ for each g and P^* is isometric). Moreover, $F(\alpha)$ is coercive, because so is $TPF(\alpha)$. □

We call $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ *divisor-left-invertible* if every square right divisor of F is left-invertible (i.e., $F = LR$, $L \in \mathcal{H}^\infty$, $R \in \mathcal{H}^\infty(\mathtt{U}) \Rightarrow SR = I$ for some $S \in \mathcal{H}^\infty(\mathtt{U})$). Note that this holds iff "the greatest square right divisor of F is I" in the sense that every square right divisor of F divides I from the right. This property lies between left-invertibility and α-weak left-invertibility, as noted in (b) and (c) below. In (a) we observe that (1) is equivalent to right coprimeness.

Theorem 2.17 (Greatest square divisor is I) *Let $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$.*

(a) *Then F is left-invertible iff every right divisor of F is left-invertible.*

(b) *If F is left-invertible, then F is divisor-left-invertible.*
The converse necessarily holds iff $\dim \mathtt{U} = \infty$ *or* $\mathtt{U} = \{0\}$ *or* $\dim \mathtt{Y} \leq \dim \mathtt{U}$.

(c) *If F is divisor-left-invertible, then F is α-weakly left-invertible.*
The converse necessarily holds iff $\dim \mathtt{U} < \infty$ *or* $\dim \mathtt{Y} < \dim \mathtt{U}$.

(d) *If F is left-invertible, then F is α-weakly left-invertible. The converse necessarily holds iff* $\mathtt{U} = \{0\}$ *or* $\dim \mathtt{Y} \leq \dim \mathtt{U} < \infty$ *or* $\dim \mathtt{Y} < \dim \mathtt{U}$.

(By $\dim \mathtt{Y} < \dim \mathtt{U}$ we mean that the (possibly infinite) cardinality of an orthonormal basis of $\mathtt{Y}$ is less than that of $\mathtt{U}$, or equivalently, that there exists a linear isometry $\mathtt{Y} \to \mathtt{U}$ but not $\mathtt{U} \to \mathtt{Y}$.)

Thus, for a $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued α-right factorization NM^{-1}, the condition "(right) $\gcd(N, M) = I$" (i.e., "every common square right divisor of N and M divides I from the right", i.e., "every common square right divisor of N and M is left-invertible") is stronger than α-weak right coprimeness (equivalent iff $\dim \mathtt{U} < \infty$) and weaker than right coprimeness (equivalent iff $\dim \mathtt{U} = \infty$ or $\mathtt{U} = \{0\}$ or $\mathtt{Y} = \{0\}$). By Theorem 2.15, this condition is a generalization of gcd-coprimeness (apparently the most natural one).

Proof of Theorem 2.17: (a) Since $F = IF$, the "if" holds. But if $GF = I$ and $F = LR$, then $(GL)R = I$, so the "only if" also holds.

(d) The first claim follows from Lemma 2.6. If $\mathtt{U} = \{0\}$, then $\mathcal{H}^\infty(\mathtt{U}) = \{0\} = \{I\}$ etc. If $\dim \mathtt{Y} < \dim \mathtt{U}$, then no $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is [α-weakly] left-invertible. The case $\dim \mathtt{U} = \dim \mathtt{Y} < \infty$ is from Corollary 2.5. The other cases have the counter-examples constructed in (b) and (c) below.

(b) The first claim is from (a). If $\dim \mathtt{U} = \infty$, then TPF (see the proof of Theorem 2.16) is a square right divisor of F, and if $QTPF = I$ for some $Q \in \mathcal{H}^\infty$, then $QTP \in \mathcal{H}^\infty$ is a left inverse of F. The other positive cases follow from (d) and Theorem 2.16.

If $1 \leq \dim \mathtt{U} < \infty$ and $\dim \mathtt{U} < \dim \mathtt{Y}$, then the F in Example 4.5(b) is not left-invertible although its square right divisors are (left-)invertible, by Theorem 2.16.

(c) The first claim and the case $\dim \mathtt{U} < \infty$ follow from Theorem 2.16 and (d). The case $\dim \mathtt{Y} < \dim \mathtt{U}$ follows the first claims in (d) and (b). If $\dim \mathtt{Y} \geq \dim \mathtt{U} = \infty$, then some $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is α-weakly left-invertible but not left-invertible, by Example 4.5(b); the TPF in the proof of Theorem 2.16 is then a non-left-invertible square right divisor of F. □

We set $F^{\mathrm{d}}(z) := F(\bar{z})^*$. Every $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ has a *dual inner-outer factorization* $F = F_o^{\mathrm{d}} F_i^{\mathrm{d}}$ (or $F^{\mathrm{d}} = F_i F_o$), where $F_i \in \mathcal{H}^\infty(\mathtt{U}_0, \mathtt{Y})$ is inner, $F_o \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{U}_0)$ is outer and $\mathtt{U}_0$ is a closed subspace of $\mathtt{U}$ [RR85] [Nik86] [Mik09]. At least when $\dim \mathtt{U} < \infty$, that factorization (and its dual) is a strictly weaker factorization than that of Theorem 2.10(a) in what comes to the left factor:

Theorem 2.18 (Outer) *If $F \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$ is α-weakly left-invertible, then F^{d} is outer.*

The converse does not hold, because the function $F(z) = z/(z+1) = F^{\mathrm{d}}(z)$ is outer but F is not α-weakly left-invertible.

Similarly, the α-w.r.c.f. is a strictly stronger tool than that provided by the dual inner-outer factorization: the former removes the common zeros of N and M also on the boundary $i\mathbb{R}$ in the sense of Theorem 2.2.

One can easily verify that F^{d} is outer iff the anti-Toeplitz operator ($\pi_- \mathscr{D}_F \pi_-$ in terms of Section 5) of F is one-to-one. But F is left-invertible iff the anti-Toeplitz operator of F is coercive [Mik09], so α-weak left-invertibility is strictly between these two conditions (at least when $\dim \mathtt{U} < \infty$).

As mentioned above, all above results on α-weak left-invertibility contain analogous results on α-weak right coprimeness. Now we go on with right-factorization-specific results.

We identify $M \in \mathcal{H}^\infty(\mathtt{U})$ with the multiplication operator $M : f \mapsto Mf$ on $\mathcal{H}^2(\mathtt{U})$. If(f) NM^{-1} is an α-w.r.c.f., then $M[\mathcal{H}^2] = D_P := \{f \in \mathcal{H}^2 \mid Pf \in \mathcal{H}^2\}$:

Theorem 2.19 (Graph) *Let $P = NM^{-1}$ be an α-right factorization and $M \in \mathcal{H}^\infty(\mathtt{U})$. Then NM^{-1} is an α-w.r.c.f. iff the graph $\left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right][D_P]$ equals $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right][\mathcal{H}^2(\mathtt{U})]$.*

This is why an α-w.r.c.f. allows one to reduce optimization problems to the stable case [Mik02].

For $P = NM^{-1}$ to be an α-right factorization, any poles (and essential singularities) of P on $\overline{\mathbb{C}^+} \cup \{\infty\}$ must be contained in M^{-1} (to have $N \in \mathcal{H}^\infty$). For N and M to be w.r.c., the function M^{-1} must not contain any other poles, i.e., the functions $N = PM$ and M may not have common zeros (this necessary condition is not sufficient for general non-rational functions). If $\mathtt{U}$ is finite-dimensional, then the poles of M^{-1} on $\mathbb{C}^+$ are isolated and hence then we can formulate that part simply:

Theorem 2.20 *If NM^{-1} is an α-w.r.c.f. and $M \in \mathcal{H}^\infty(\mathbb{C}^n)$, $n \in \mathbb{N}$, then the nonremovable singularities of M^{-1} on $\mathbb{C}^+$ are the same as those of $P := NM^{-1}$.*

As illustrated above, M is the function that "stabilizes P with minimum effort" (i.e., $N := PM$ is bounded on $\mathbb{C}^+$ but M is as close to the identity as possible, in the sense of the last paragraph of Theorem 2.1).

Theorem 2.20 is a special case (by Theorem 2.2) of the following fact that the maximal connected domains of P and M^{-1} are the same wherever $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](z)$ is coercive.

Theorem 2.21 *If NM^{-1} is an α-w.r.c.f., and $P := NM^{-1}$ has a holomorphic extension to some open and connected $\Omega \subset \mathbb{C}^+$, then M is invertible at those $z \in \Omega$ for which $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](z)$ is coercive.*

(The coercivity assumption is not redundant, by Example 4.6. However, it is superfluous at least if the nonconstant part of M is sufficiently compact, by Lemma 7.8.)

Proof: Let Ω' be the (necessarily open) connected component of $\{z \in \mathbb{C}^+ \mid M(z) \in \mathcal{GB}(\mathtt{U})\}$ that contains α. We assume the existence of some $z \in \partial\Omega' \cap \Omega$ and obtain a contradiction; this proves the theorem.

Pick $f \in \mathcal{H}^2(\mathbb{C})$ such that $f(z) = 1$. Since $M(z)$ is not invertible, it is not coercive, hence there exist $\{u_n\} \subset \mathtt{U}$ such that $\|u_n\| = 1$ $\forall n$ and $M(z)u_n \to 0$, as $n \to \infty$. But $g_n := Mfu_n \in D_P$ (see Theorem 2.19) and $g_n(z) = M(z)u_n \to 0$, hence $(Pg_n)(z) = N(z)u_n \not\to 0$, because $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](z)$ is coercive. Yet $(Pg_n)(z) = P(z)g_n(z) \to 0$, a contradiction. □

Note that in Theorems 2.21 we could replace $\mathbb{C}^+$ by any open, connected $\Omega'' \subset \mathbb{C}$ such that N and M have holomorphic extensions to Ω'' (if we set $P := NM^{-1}$ wherever applicable).

Lemma 2.22 ($\mathbf{P+P_0=NM^{-1}}$) *If $P = NM^{-1}$ is a $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued α-[w.]r.c.f. and $P_0 \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$, then $P + P_0 = (N + P_0M)M^{-1}$ is an α-[w.]r.c.f.*

Proof: If $(N + P_0M)f, Mf \in \mathcal{H}^2$, then $Nf = (N + P_0M)f - P_0Mf \in \mathcal{H}^2$, hence then $f \in \mathcal{H}^2$. If $\tilde{X}M - \tilde{Y}N = I$, then $(\tilde{X} + \tilde{Y}P_0)M - \tilde{Y}(N + P_0M) = I$. □

In the matrix-valued case, right and left factorizations coexist:

Lemma 2.23 *If a $\mathcal{B}(\mathbb{C}^n,\mathtt{Y})$-valued function P has an α-right factorization ($\alpha \in \mathbb{D}$), then P has an α-left factorization.*

Hence then P has a α-w.r.c.f. and a "α-w.l.c.f.", by Theorem 1.1 (and its dual). We do not know whether this holds for $\mathtt{U}$ in place of $\mathbb{C}^n$.

Proof: Let $P = NM^{-1}$ be an α-right factorization and set $f := \det M \in \mathcal{H}^\infty(\mathbb{C})$. Then $M^{-1} = f^{-1}F$ for $F := \operatorname{adj}(M) \in \mathcal{H}^\infty(\mathtt{U})$, hence $P = \tilde{M}^{-1}\tilde{N}$ is an α-left factorization, where $\tilde{N} := NF$, $\tilde{M} := fI$. (Indeed, $\det \tilde{M}(\alpha) = f(\alpha)^n \neq 0$.) □

The coercivity assumption on $F(\alpha)$ seems somewhat artificial (and it has not been used explicitly before this article). For an α-right factorization, it is redundant, but in general it is needed to avoid labeling the function $F(z) := (\alpha - z)/(\alpha + z)$ as α-weakly left-invertible. However, even if we dropped this requirement, most results would still hold:

Remark 2.24 Redefine α-weak left-invertibility by dropping the coercivity requirement. Then all above results hold (with the same proofs) except that Corollaries 2.5 and 2.13, Theorems 2.7, 2.10(b), 2.15 and 2.18, and the last claims of Theorems 2.16 and 2.17(c)&(d) become false and in Theorem 2.2 we must require that $z \neq \alpha$. ◁

The function $F(z) := \frac{\alpha - z}{\alpha + z}$ is α-weakly left-invertible in this alternative sense but not in the original sense.

Related results for "proper" weak left-invertibility are given in Remark 3.6. Further results on this weaker concept are given in [Mik02, Chapter 4 and Sections 6.4–6.5] under the name "quasi-left-invertibility". Note that α-weak coprimeness and "α-quasi coprimeness" of N and M are equivalent when M^{-1} is α-proper.

Alternative, independent proofs of parts of Theorems 2.1, 2.14, 2.15, and 2.16 can be based on [Qua06] combined with [Qua05], in the matrix-valued case; moreover, the matrix-valued Theorem 2.19 was given in [GS93] except for the fact that that M can be chosen to be square. Lemma 2.22 is from [Mik02]. More details on the results of this section (in the discrete-time context) are given in [Mik07g, Section 6].

3 Proper weak left-invertibility and w.r.c.f.'s

In this section we establish the results of Section 2 for (proper) w.r.c.f.'s (i.e., we "remove the α's") and derive further results on w.r.c.f.'s.

If NM^{-1} is a $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued [$\alpha$-]r.c.f., then any $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] \in \mathcal{G}\mathcal{H}^\infty(\mathtt{Y}\times\mathtt{U})$ is called a *[α-]d.c.f.* (doubly coprime factorization).

In a [w.]r.c.f. the denominator M is required to be boundedly invertible on a half-plane, whereas in an α-[w.]r.c.f. invertibility is only required at α. Fortunately, any such factorization of a proper function is unique (up to an element in $\mathcal{G}\mathcal{H}^\infty$):

Theorem 3.1 ($\alpha \Leftrightarrow$ proper) *Let $\omega \geq 0$, $\alpha \in \mathbb{C}^+_\omega$ and $P \in \mathcal{H}^\infty_\omega(\mathtt{U},\mathtt{Y})$.*

(a) *Then any α-r.c.f. of P is a r.c.f. of P, and any r.c.f. of P is an α-r.c.f. of P. (The same holds with "d.c.f." in place of "r.c.f.".)*

(b) *If $P = NM^{-1}$ is a right factorization and $M \in \mathcal{G}\mathcal{H}^\infty_\omega$, then any α-w.r.c.f. of P is a w.r.c.f. and any w.r.c.f. of P is an α-w.r.c.f.*

(c) *If $\dim \mathtt{U} < \infty$, then a right factorization $P = NM^{-1}$ is a w.r.c.f. iff N and M are gcd-coprime.*

For nonproper functions the claims (a) and (b) do not hold (see Example 4.2 for (b)). If $\dim \mathtt{U} = \infty$, then (c) does not hold and no w.r.c. N and M have $\gcd(N, M) = I$ (since $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] = (\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] L)R$ with $L, R \in \mathcal{B}(\mathtt{U})$, $LR = I$, $R \notin \mathcal{GB}(\mathtt{U})$).

Proof of Theorem 3.1: (a) Let $P = NM^{-1}$ be a r.c.f. of P with $X, Y \in \mathcal{H}^\infty$, $XM + YN \equiv I$. Then $M^{-1} = X + YNM^{-1} = X + YP$, which is bounded on $\mathbb{C}^+_\omega$, hence $P = NM^{-1}$ is an α-r.c.f. The converse is analogous, and so is the d.c.f. case.

(b) If $P = NM^{-1}$ is a right factorization and $M \in \mathcal{GH}^\infty_\omega$, then $P = NM^{-1}$ is an α-right factorization, hence then P has a w.r.c.f. $P = N_0 M_0^{-1}$ and an α-w.r.c.f. $P = N_1 M_1^{-1}$. By Theorem 2.8, $M_1 \in \mathcal{GH}^\infty_\omega$ and $N_1 M_1^{-1}$ is a z-w.r.c.f. for any $z \in \mathbb{C}^+_\omega$, hence $N_1 M_1^{-1}$ is a w.r.c.f., hence $M_1 = MV$ for some $V \in \mathcal{G}\mathcal{H}^\infty$.

(c) This follows from (b) and Theorem 2.15. □

We call $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ *weakly left-invertible* if

1. $F^*F \geq \epsilon I$ on $\mathbb{C}^+_\omega$ for some $\omega \geq 0$ and $\epsilon > 0$, and
2. $Ff \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$ for every proper $\mathtt{U}$-valued function f (i.e., for every $f \in \mathcal{H}^\infty_\infty(\mathbb{C}, \mathtt{U})$).

We call $N, M \in \mathcal{H}^\infty$ *w.r.c.* if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is weakly left-invertible; note that here condition "1." is redundant if NM^{-1} is a right factorization.

The equivalence of w.r.c.f.'s and α-w.r.c.f.'s (and gcd-w.r.c.f.'s) was established in Theorem 3.1(b), but an α-w.r.c. pair need not be w.r.c. by Example 4.2 (where the condition "1." is not satisfied by $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$).

Almost all of our α-w.r.c.f. results hold for (proper) w.r.c.f.'s too:

Theorem 3.2 (Non-α w.r.c.f.'s) *The results 2.1–2.9 and 2.16–2.23 also hold when we remove each "$\alpha-$" and in Corollary 2.4 we replace "$F(\alpha) \in \mathcal{GB}(\mathtt{U}, \mathtt{Y})$" by "$F \in \mathcal{GH}^\infty_\infty$" and in Theorem 2.8 we replace "$\alpha \in \Omega$" by "Ω contains a right half-plane".*

Note that here Theorem 2.1 becomes Theorem 1.1. Theorem 1.1 could, alternatively, be proved using [Mik06b] (and Lemma 8.1); also the rest of Theorem 3.2 could be proved directly in continuous-time using [Mik06b].

Proof: 1° Theorem 1.1 follows from Theorems 3.1 and 2.1 (e.g., if $P = N_1 M_1^{-1}$ is a right factorization and $P = NM^{-1}$ a w.r.c.f., then $V := M^{-1}M_1 \in \mathcal{GH}^\infty_\infty$).

2° If F is weakly left-invertible and R is proper, then there exists $\alpha \in \mathbb{C}^+$ such that F is α-weakly left-invertible and R is α-proper, hence each "α-" in Lemma 2.3 and Theorem 2.2 can be removed.

3° Now we prove Theorem 3.3 before going on. If F is weakly left-invertible and $F = F_w F_r$ as in Lemma 2.11 (with $F_r - 1$ being proper), then $F_r^{-1} \in \mathcal{H}^\infty$, by the non-$\alpha$-version of Lemma 2.3, hence then F is α-weakly left-invertible (because so is F_w). For the converse, assume that F is α-weakly left-invertible. Any $f \in \mathcal{H}^\infty_\infty$ is defined at some $\beta \in \mathbb{C}^+_\omega$ and F is β-weakly left-invertible, by Corollary 2.12, hence $Ff \in \mathcal{H}^2$ implies that $f \in \mathcal{H}^2$; because f was arbitrary, F is weakly left-invertible.

4° The remaining results now follow from the original ones using Theorems 3.3 and 3.1 (in Theorem 2.7 use the fact that (i) trivially implies (iii)) with the three exceptions listed below.

5° *Theorems 2.16 & 2.17:* the original proofs (mutatis mutandis) yield the claims.

6° *Lemma 2.23:* If $P \in \mathcal{H}^\infty_\omega$ has a right factorization, then P^{d} has an α-w.r.c.f. for some $\alpha \in \mathbb{C}^+_\omega$, by Lemma 2.23 and Theorem 1.1; by Theorem 3.1(b), that factorization is a w.r.c.f. of P^{d} (whose dual is a left factorization (even w.l.c.f.) of P). □

See Corollary 3.5 (resp., Theorem 3.3, Remark 3.6) for the "non-α forms" of Theorem 2.10 (resp., Corollary 2.12, Remark 2.24). As described below, the "non-α forms" of Theorems 2.14 and 2.15 are not true in general but they are true for $F = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ with M^{-1} proper.

Under the condition "1.", proper and α-weak left-invertibility are equivalent:

Theorem 3.3 ($\alpha \Leftrightarrow$ proper) *Let $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$, $\omega \geq 0$, $\alpha \in \mathbb{C}^+_\omega$, $\epsilon > 0$ and $F^*F \geq \epsilon I$ on $\mathbb{C}^+_\omega$. Then F is weakly left-invertible iff F is α-weakly left-invertible.*

Thus, weak left-invertibility implies α-weak left-invertibility on a right half-plane.

(This was shown in the proof of Theorem 3.2.)

Note that, in the matrix-valued case, weak left-invertibility implies α-weak left-invertibility for every $\alpha \in \mathbb{C}^+$, by Theorem 3.3 and Corollary 2.13, but not in the operator-valued case, by Example 4.6. Recall from Theorems 2.14–2.17 that α-weak left-invertibility, irreducibility, divisor-left-invertibility and the property of Theorem 2.14 are all equivalent (and hence strictly weaker than weak left-invertibility, by Example 4.2, though equivalent to it under the assumptions of Theorem 3.3) in the matrix-valued case.

Note from Corollary 2.4 that if $F \in \mathcal{H}^\infty(\mathtt{U})$ is weakly left-invertible, α, ω are as above and $F(\alpha) \in \mathcal{GB}(\mathtt{U})$, then $F \in \mathcal{G}\,\mathcal{H}^\infty(\mathtt{U})$.

In condition "2." it suffices to consider merely the functions $f \in \mathcal{H}^2_\infty(\mathtt{U})$:

Theorem 3.4 *Let $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$. Then the implication $Ff \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$ is true for every $f \in \mathcal{H}^\infty_\infty(\mathbb{C}, \mathtt{U})$ iff it is true for every $f \in \mathcal{H}^2_\infty(\mathtt{U})$.*

Proof of Theorem 3.4: "Only if" holds because $\mathcal{H}^2_\infty \subset \mathcal{H}^\infty_\infty$ (by [HP57, Theorem 6.4.2]), so suppose that "$Ff \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$" holds for every $f \in \mathcal{H}^2_\infty(\mathtt{U})$ and that $f \in \mathcal{H}^\infty_\infty$ and $Ff \in \mathcal{H}^2$.

Set $g_n := ns^{-1}(1 - \mathrm{e}^{-s/n})$. Then $g_n = n\widehat{\chi_{[0,1/n)}}$, hence $\|g_n\|_\infty \leq 1$ and $g_n \in \mathcal{H}^2(\mathbb{C}) \cap \mathcal{H}^\infty$. Consequently, $\|Fg_nf\|_2 \leq \|Ff\|_2 =: K < \infty$, hence $g_nf \in \mathcal{H}^2(\mathtt{U})$ (because $g_nf \in \mathcal{H}^2_\infty$) for every n. In particular, $f = g_n^{-1} \cdot (g_nf)$ is holomorphic $\mathbb{C}^+ \to \mathtt{U}$.

Let $\epsilon > 0$ be as in Theorem 2.2. Then $\|g_n f\|_2 \le K/\epsilon \ \forall n$. But $g_n(s) \to 1$, as $n \to +\infty$, for each $s \in \mathbb{C}^+$. By the Dominated Convergence Theorem,

$$\int_{-T}^{T} \|f(\omega+it)\|_{\mathtt{U}}^2 \, dt = \lim_{n\to\infty} \int_{-T}^{T} \|(g_n f)(\omega+it)\|_{\mathtt{U}}^2 \, dt \le \limsup_{n\to\infty} \|g_n f\|_{\mathcal{H}^2}^2 \le (K/\epsilon)^2 \tag{3}$$

for each $T > 0$ and $\omega > 0$, hence $\|f\|_2 \le K/\epsilon$, hence $f \in \mathcal{H}^2(\mathtt{U})$. □

The function $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ of Example 4.2 illustrates that the (proper) weak-left-invertibility-analogy of Theorem 2.10(a) requires an additional assumption. That is made below.

Corollary 3.5 *If $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ and $F^* F \ge \epsilon^2 I$ on $\mathbb{C}^+_\omega$ for some $\omega \ge 0$ and $\epsilon > 0$, then $F = F_w F_r$, where $F_r \in \mathcal{H}^\infty(\mathtt{U}) \cap \mathcal{G}\mathcal{H}^\infty_\omega(\mathtt{U})$ and $F_w \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is inner and z-weakly left-invertible for each $z \in \mathbb{C}^+_\omega$, hence weakly left-invertible.*

(This follows from Lemma 2.11.)

We make a remark on a weaker variant of weak left-invertibility.

Remark 3.6 ($F^* F \ge \epsilon I$ on $\mathbb{C}^+_\omega$) Replace the condition "1." in the definition of weak left-invertibility by "F is coercive on $\in \mathbb{C}^+_\omega$ for some $\omega \ge 0$, and". By Example 4.1 below, this new property does not imply the old property, not even α-weak left-invertibility (for any $\alpha \in \mathbb{C}^+_\omega$; not even the weaker form described in Remark 2.24).

The example $F = \mathrm{e}^{-\cdot} I$ also shows that with this new definition, in Theorem 3.2, we would have to omit Corollary 2.5 and the part "or $\dim \mathtt{Y} \le \dim \mathtt{U} < \infty$" in Theorem 2.17(d). However, otherwise Theorem 3.2 would remain true (even if we would completely remove "1."). This can be shown by modifying the original proofs. ◁

Recall that "1." is redundant for right factorizations, hence for them all forms of weak coprimeness are equivalent including that of Remark 3.6 above.

Notes: For comparison, we note that the concept "weakly right-prime" in [Qua03] in a matrix-valued $\mathcal{H}^\infty$ context can be shown equivalent to "weakly α-left-invertible times a constant surjective matrix" (use Theorem 2.15). Moreover, "weakly right-prime" is also equivalent to "quasi-left-invertible times a constant surjective matrix", where quasi-left-invertibility is defined as in Remark 2.24. In, e.g., [Qua06], w.r.c.f.'s are defined as in [Smi89, Lemma 4] (in the $\mathcal{H}^\infty$ context).

4 Counter-examples

We present here examples that show it impossible to extend or reverse certain implications.

By Theorem 2.2, [α-]weakly left-invertible maps have no common zeros on $\mathbb{C}^+$ nor "a.e." on the boundary. However, the converse is not true:

Example 4.1 (No zeros $\not\Rightarrow$ α-w.r.c.) Let $F(s) = \mathrm{e}^{-s}$. Then F is inner but not weakly left-invertible, not even α-weakly left-invertible for any $\alpha \in \mathbb{C}^+$, because $f(s) := \mathrm{e}^s/(s+1) \notin \mathcal{H}^2$ but $Ff \in \mathcal{H}^2$. Moreover, F is continuous on $\overline{\mathbb{C}^+}$ and coercive on $\mathbb{C}^+$ but not uniformly coercive: $|F(s)| = \mathrm{e}^{-\operatorname{Re} s} \to 0$, as $s \to +\infty$. Yet we have $Ff \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$ for each proper scalar function f. $\triangleleft$

Naturally, F and 0 are then not even α-w.r.c., although they do not have common zeros (outside $+\infty$) and F is inner.

Proof: If $g \in \mathcal{H}^2_\omega$ for some $\omega \geq 0$, then $g = \mathrm{e}^{\cdot} Fg$, hence $\|g(r + i\cdot)\|_{\mathrm{L}^2} = \mathrm{e}^r\|(Fg)(r+i\cdot)\|_{\mathrm{L}^2} \leq \mathrm{e}^r\|Fg\|_{\mathcal{H}^2}$ $(r > 0)$, hence $\|g\|_{\mathcal{H}^2} \leq \max\{\|g\|_{\mathcal{H}^2_\omega}, \mathrm{e}^\omega\|Fg\|_{\mathcal{H}^2}\}$. Therefore, we have $Fg \in \mathcal{H}^2 \implies g \in \mathcal{H}^2$ for each proper scalar function g, by Theorem 3.4. □

By Theorem 3.1, weak, α-weak and gcd-coprimeness are equivalent for a right factorization. However, for general functions weak coprimeness is strictly stronger than α- or gcd-coprimeness:

Example 4.2 (Scalar α-w.r.c. $\not\Rightarrow$ w.r.c.) The functions $N(s) := s\mathrm{e}^{-s}/(s+1)$ and $M(s) := 1/(s+1)$ are gcd-coprime [Smi89], i.e., α-w.r.c. for every $\alpha \in \mathbb{C}^+$ (Theorem 2.15), but yet not w.r.c., because $N(+\infty) = 0 = M(+\infty)$. $\triangleleft$

By the substitution $s \mapsto 1/s$, we obtain a scalar-valued w.r.c.f. that is not a r.c.f.:

Example 4.3 (Scalar w.r.c.f. $\not\Rightarrow$ r.c.f.) The functions $N(s) := \mathrm{e}^{-1/s}/(s+1)$ and $M(s) := s/(s+1)$ form a w.r.c.f. but not a r.c.f., because $N(0+) = 0 = M(0)$. $\triangleleft$

(Now $M \in \mathcal{GH}^\infty_\infty$, so Theorem 3.1(b), Example 4.2, and Lemma 7.3 imply that NM^{-1} is a w.r.c.f.)

As mentioned in the introduction, the w.r.c.f. determined by the LQ-optimal feedback need not be Bézout coprime:

Example 4.4 (LQ-optimal feedback is not coprime) **(a)** By Theorem 1.2, the function $P := NM^{-1}$ of Example 4.3 has an output-stabilizable (SISO) realization. The LQ-optimal state-feedback for this realization determines a w.r.c.f. $N_1M_1^{-1}$ of P, by Theorem 1.2, but that w.r.c.f. is not a r.c.f., because P does not have an r.c.f., by the last claim in Theorem 1.1.

(b) If, in (a), we use Example 4.6 in place of Example 4.3, then we have $\left[\begin{smallmatrix} N_1 \\ M_1 \end{smallmatrix}\right](1)$ noncoercive, because $\left[\begin{smallmatrix} N_1 \\ M_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] V$ for some $V \in \mathcal{GH}^\infty$, by Theorem 1.1. $\triangleleft$

Naturally, Example 4.3 can be extended to higher dimensions:

Example 4.5 (General w.r.c.f. $\not\Rightarrow$ r.c.f.) **(a)** If $\dim \mathtt{U} \geq 1$ and $\dim \mathtt{Y} \geq 1$, then there exists a $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued w.r.c.f. NM^{-1} that is not a r.c.f.

(b) Consequently, if $1 \leq \dim \mathtt{U} < \dim \mathtt{Y}$ or $\infty = \dim \mathtt{U} \leq \dim \mathtt{Y}$, then there exists $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ that is weakly left-invertible but not left-invertible. $\triangleleft$

(The above dimensionality assumptions are also necessary, by Theorem 2.16.)

Proof: (a) Set $M := \left[\begin{smallmatrix} M' & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathbb{C}\times\mathtt{U}_2)$, $N := \left[\begin{smallmatrix} N' & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathbb{C}\times\mathtt{U}_2, \mathbb{C}\times\mathtt{Y}_2)$, where N' and M' are the N and M of Example 4.3 and $\mathtt{U} = \mathbb{C}\times\mathtt{U}_2$, $\mathtt{Y} = \mathbb{C}\times\mathtt{Y}_2$.

(b) There exists $T \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ such that $T^*T = I$ and T is not onto (this is elementary [Mik02, Lemma A.3.1(a4)]). Set $\mathtt{Y}_1 := \operatorname{Ran}(T)$, $\mathtt{Y}_2 := \mathtt{Y}_1^\perp$. Choose N and M as in (a) with $\mathtt{Y}_2$ in place of $\mathtt{Y}$ and set $F := \left[\begin{smallmatrix} N \\ TM \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & T \end{smallmatrix}\right]\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$. □

If we replace Example 4.3 by Example 4.2 in the above proof, we obtain in (a) an α-w.r.c. pair N, M that is not w.r.c. and in (b) an α-weakly left-invertible function that is not weakly left-invertible.

By Theorem 2.2, $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is injective on $\mathbb{C}^+$ for any w.r.c.f. NM^{-1}. It need not be coercive on $\mathbb{C}^+$:

Example 4.6 (Continuous w.r.c.f $\not\Rightarrow$ r.c.f.) Let $\mathtt{U} := \ell^2(\mathbb{N}) =: \mathtt{Y}$. There exists a normalized w.r.c.f. NM^{-1} such that $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](1)e_k \to 0$, as $k \to \infty$ (hence $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](1)$ is not coercive and N and M are not r.c.) but N and M are continuous $\overline{\mathbb{C}^+} \cup \{\infty\} \to \mathcal{B}(\mathtt{U})$ and $\mathcal{H}^\infty_\omega$ for every $\omega > -1$.

Moreover, (ii) of Theorem 2.7 is satisfied and $P := NM^{-1}$ does not have an α-w.r.c.f. for any $\alpha \in \{1\} \cup \{1 - i/2k \mid k \in \mathbb{N}\}$ but NM^{-1} is an α-w.r.c.f. for every other $\alpha \in \mathbb{C}^+$. ◁

(Note that here N and M belong to the "half-plane algebra" (disc algebra), even to the Wiener class.)

Proof: (Here we make N and M inner, so, to be normalized, they should be divided by $\sqrt{2}$.)

1° Set $a_k := 1 + i/2k$, $b_k := 1 - i/2k$ ($k \in \mathbb{N}$), so that $a_k \to 1$ and $b_k \to 1$, as $k \to \infty$. Set $N_k(s) := (s - a_k)/(s + a_k)$, $M_k(s) := (s - b_k)/(s + b_k)$, and let $N, M \in \mathcal{H}^\infty(\mathtt{U})$ be the (diagonal) inner functions determined by $N(s)e_k = N_k(s)e_k$, $M(s)e_k = M_k(s)e_k$ ($s \in \mathbb{C}^+$, $k \in \mathbb{N}$), where $e_k := \chi_{\{k\}}$. Then $\|N_k(1)\| \le 1/2k \to 0$ and $\|M_k(1)\| \to 0$, as $k \to \infty$, hence $\|\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](1)e_k\| \to 0$ (hence N and M are not r.c., by Lemma 2.6(a)). Moreover, $N, M \in \mathcal{G}\mathcal{H}^\infty_\omega$ for any $\omega > 1$, so NM^{-1} is a right factorization.

2° But N_k and M_k are r.c., by Theorem 2.7. Thus, if $\Omega \subset \mathbb{C}^+$ and $f : \Omega \to \mathtt{U}$ satisfies $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f \in \mathcal{H}^2$, then $f_k \in \mathcal{H}^2$ ($k \in \mathbb{N}$), where f_k is the kth component of f, and $\|\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f_k e_k\|_2^2 = 2\|f_k\|_2^2$ ($k \in \mathbb{N}$), hence $\|\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f\|_2^2 = 2\|f\|_2^2$, hence $f \in \mathcal{H}^2$. In particular, N and M are α-w.r.c. for every $\alpha \in \mathbb{C}^+ \setminus \{1\}$, because $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](\alpha)$ is coercive for those α, by 3° below. By 1°, N and M are not 1-w.r.c.

3° Set $B := \{1\} \cup \{b_k \mid k \in \mathbb{N}\}$. One easily verifies that $M(z), N(\bar{z}) \in \mathcal{G}\mathcal{B}(\mathtt{U})$ $\forall z \in \mathbb{C}^+ \setminus B$.

4° Since $N_k = 1 - 2a_k\widehat{\mathrm{e}^{-a_k\cdot}}$, we observe that $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] + \widehat{f}$, where $f \in \mathrm{L}^1(\mathbb{R}_+; \mathcal{B}(\mathtt{U}, \mathtt{U}\times\mathtt{U}))$, because $\|f(t)\|_{\mathcal{B}(\mathtt{U})} = \sqrt{2}\cdot 2 \cdot |1 + i/2|\mathrm{e}^{-t}$ ($t \ge 0$). In particular, N and M are continuous $\overline{\mathbb{C}^+} \cup \{\infty\} \to \mathcal{B}(\mathtt{U})$ and $\mathcal{H}^\infty_\omega$ for every $\omega > -1$. □

In fact, given, e.g., any compact disc $D \subset \mathbb{C}^+$, there exists a w.r.c.f. NM^{-1} (with separable input and output spaces) such that $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](z)$ is not coercive for any $z \in D$ but N and M are z-w.r.c. for all $z \in \mathbb{C}^+ \setminus D$. See the comments below [Mik07g, Example 7.3] for a sketch.

If NM^{-1} is a w.r.c.f., then every common right square factor of N and M is weakly left-invertible (hence invertible if $\dim \mathtt{U} < \infty$), by Theorem 2.16. In general it need not be left-invertible (nor right-invertible):

Example 4.7 (divisor not left-invertible) Let $N, M, \mathtt{U}, \mathtt{Y}$ be as in Example 4.6. Then $F := J^{-1}\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathtt{U})$ is weakly left-invertible for any $J \in \mathcal{GB}(\mathtt{U}, \mathtt{Y} \times \mathtt{U})$. However, $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] = JF$, but F is not left-invertible; in fact, $F(1) \in \mathcal{B}(\mathtt{U})$ is not left-invertible (nor right-invertible). Nevertheless, $F(z)$ is left-invertible for every $z \in \mathbb{C}^+ \setminus \{1\}$. ◁

By B_G we denote the Blaschke product formed with the zeros of G. We soon need the following:

Lemma 4.8 *Let $F, G \in \mathcal{H}^\infty(\mathbb{C})$ have no common zeros and $G \not\equiv 0$. Let $\Omega \subset \mathbb{C}^+$ be open and let $f : \Omega \to \mathbb{C}$ be such that $Ff, Gf \in \mathcal{H}^2$. Then f has a holomorphic extension $f : \mathbb{C}^+ \to \mathbb{C}$ and $fG/B_G \in \mathcal{H}^2$.*

Thus, if $G = B_G$, then F and G are w.r.c.

Proof: If $F \equiv 0$, then $g := G^{-1}Gf$ is a holomorphic extension of f. If $F \not\equiv 0$, then g and $h := F^{-1}Ff$ are meromorphic extensions of f with no common singularities (on $\mathbb{C}^+$), hence they coincide outside their singularities, hence their singularities are removable, and we get a holomorphic extension $\mathbb{C}^+ \to \mathbb{C}$ of f.

Consequently, $|B_{Gf}| \le |B_G|$ (because B_{Gf} has at least the same terms as B_G), hence $|Gf/B_G| \le |Gf/B_{Gf}|$. But $Gf/B_{Gf} \in \mathcal{H}^2$, by [Hof88, pp. 132–133], hence $Gf/B_G \in \mathcal{H}^2$. □

Now we can construct an *exponentially stable* (which means that $M, N \in \mathcal{H}^\infty_\omega$ for some $\omega < 0$) scalar non-r.c. w.r.c.f.:

Example 4.9 (Exponential w.r.c.f.$\not\Rightarrow$r.c.f.) Choose any $a > 0$ such that we have $\sum_{k=1}^\infty |2/(s + 1 - ik^2 + a)| < \log 3/2$ for all $s \in \mathbb{C}^+$. Let G and F be the Blaschke products with zeros $a + ik^2$ and $a + ik^2 + i/k$ ($k = 1, 2, \ldots$), respectively.

Then the functions $M := G(1 + \cdot)$, $N := F(1 + \cdot)$ form a w.r.c.f., but yet not an r.c.f., hence the map NM^{-1} does not have a r.c.f., by Theorem 1.1. Moreover, $N, M \in \mathcal{H}^\infty_{-1}(\mathbb{C})$ and N and M are α-w.r.c. for every $\alpha \in \mathbb{C}^+$. ◁

Proof: 1° The choice of a is possible, because $\sum_{k=K}^\infty < \log 3/2$ for some K (the terms are $\le 2k^{-2}$) and $\sum_{k=1}^{K-1}$ can be made arbitrarily small by increasing a (that does not increase any other term either).

2° By Lemma 4.8, F and G are w.r.c. Set $\beta_k := a + ik^2$, $\alpha_k := a + ik^2 + 1/k$. Since $|G(\alpha_n)| = \prod_k |\alpha_n - \beta_k|/|\alpha_n + \bar{\beta}_k|$, where the terms are ≤ 1 and $|\alpha_n - \beta_n|/|\alpha_n + \bar{\beta}_n| < 1/n$, we observe that $|G(\alpha_n)| < 1/n \to 0$, as $n \to +\infty$. Since $F(\alpha_n) = 0$, by definition, N and M are not r.c., by Lemma 2.6(a).

3° *We have* $M^{-1}, N^{-1} \in \mathcal{H}^\infty_\infty$: Now $|G(s)| = \prod_{k=1}^\infty |1 - a_k(s)|$, where $a_k(s) := 2a/(s + a - ik^2)$. In the spirit of 1° one can show that for some $\omega > a$ we have, for every $s \in \mathbb{C}^+_\omega$, that $|a_k(s)| < 1$, and $\sum_k |a_k(s)| \le \log 3/2$. By [Rud87, Lemma 15.3], $||G(s)| - 1| \le 3/2 - 1$, hence $|G(s)| \ge 1/2$ on $\mathbb{C}^+_\omega$, hence $G^{-1} \in \mathcal{H}^\infty_\infty$, hence $M^{-1} \in \mathcal{H}^\infty_\infty$. Similarly, $N^{-1} \in \mathcal{H}^\infty_\infty$.

4° Let $\Omega \subset \mathbb{C}^+$ be open and let $f : \Omega \to \mathbb{C}$ be such that $Nf, Mf \in \mathcal{H}^2$. By Lemma 4.8, $Mf/B \in \mathcal{H}^2$, where $B := B_M$. By a direct computation, we get $|M(s)|/|B(s)| = \prod_k |1 - 2/(s + 1 - ik^2 + a)|$, hence $|M|/|B| \ge 1/2$, as in 3°. Consequently, $M/B \in \mathcal{G}\mathcal{H}^\infty$, hence $f \in \mathcal{H}^2$. In particular, N and M are w.r.c. and α-w.r.c. for every $\alpha \in \mathbb{C}^+$. □

For future use in counter-examples on stabilizability, we make here some remarks that are obvious from the proof of Example 4.9:

Remark 4.10 *For any $\omega \ge 0$, the functions $N(\cdot + \omega)$ and $M(\cdot + \omega)$ in Example 4.9 form a w.r.c.f. of $P(\cdot + \omega)$. If $\omega \le a - 1$, then that w.r.c.f. is not a r.c.f., hence then $P(\cdot + \omega)$ does not have an r.c.f. Moreover, the a in Example 4.9 can be arbitrarily increased. Finally, as in 3°, we can find some $R > 0$ such that $|G(s)| > 1/2$ when $s \in \mathbb{C}^+$ and $\operatorname{Im} s \le -R$.*

Note also that, given $R \in \mathbb{R}$, any functions $N_1, M_1 \in \mathcal{H}^\infty$ are [w.]r.c. iff $N_1(\cdot + iR), M_1(\cdot + iR)$ are [w.]r.c.

Example 4.9 determines a w.r.c.f. NM^{-1}. From the proof of [Mik02, Lemma 6.6.29] (or from that of [Sta05, Theorem 8.4.6(iii)]) we observe that $P := NM^{-1}$ has an "exponentially stabilizable" realization. (Analogously, it has an exponentially detectable realization too, but a single realization of P cannot be both stabilizable and detectable, by Theorem 1.3.) Thus, the ("exponentially stable") w.r.c.f. determined by the LQ-optimal state feedback for that realization is not a r.c.f.

A further treatment of the subject is given in [Mik07d], where also many control-theoretic consequences of the other above examples are derived and discussed. Some related discussion is given at the end of [Mik07g, Section 4] for discrete-time systems. There it is also noted that no discrete-time counterpart of Example 4.9 exists, because discrete-time "power stable" w.r.c.f.'s are r.c.f.'s.

Example 4.3 is adapted from [Smi89], where a first example of a non-r.c. w.r.c. pair was given. The first example based on definition (2) was due to Sergei Treil (personal communication, before the equivalence Theorem 2.15).

The other examples in this section are new. Further "counter-examples" are given in the previous sections below certain results.

5 Introductory theorems and WPLSs

In this section we prove and further explain the fundamental Theorems 1.2 and 1.3.

Before that we very briefly recall the basics of WPLSs and state feedback. Further details can be found in, e.g., [Sta98a], [Sta05], [SW02], [Mik02], [Mik06b], [Mik07d] and [Mik06a] in the notation of Definition 5.1, and in different but equivalent notation in, e.g., [Sal87], [Sal89], [Wei94b], [Wei94a], [WR00], [Sta98a], [Sta98b] and [OC04]. However, most of this article, excluding some of the proofs, can be understood without any knowledge on WPLSs, so a busy reader can skip this section.

We first introduce the WPLS through their most well-known special case, namely that having bounded input and output operators (B and C). Set $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{R}_- := (-\infty, 0)$. Let A generate a strongly continuous semigroup $\mathscr{A}$ on $\mathtt{X}$, and let $B \in \mathcal{B}(\mathtt{U}, \mathtt{X})$, $C \in \mathcal{B}(\mathtt{X}, \mathtt{Y})$ and $D \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$. Given an input function $u : \mathbb{R}_+ \to \mathtt{U}$ and initial state $x_0 \in \mathtt{X}$, we define the state trajectory $x : \mathbb{R}_+ \to \mathtt{X}$ and output function $y : \mathbb{R}_+ \to \mathtt{Y}$ through $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, $x(0) = x_0$.

Let $\mathscr{D}$ denote that map $u \mapsto y$, and set $\mathscr{C} := C\mathscr{A}$ (i.e., $(\mathscr{C}x_0)(t) = C\mathscr{A}^t x_0$ for $x_0 \in \mathtt{X},\ t \ge 0$), and $\mathscr{B}v := \int_0^\infty \mathscr{A}^r Bv(-r)\,dt$ for suitable $v : \mathbb{R}_- \to \mathtt{U}$. Then $y = \mathscr{C}x_0 + \mathscr{D}u$, and the above four maps constitute a WPLS $\Sigma = \left[\begin{smallmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{smallmatrix}\right]$ (Definition 5.1). We call $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ the generators of Σ. The dual system $\Sigma^{\rm d}$ is the one having generators $\left[\begin{smallmatrix} A^* & B^* \\ C^* & D^* \end{smallmatrix}\right]$, and the transfer function of Σ equals $\hat{\mathscr{D}}(s) := D + C(s - A^{-1})B$.

From this illustrative special case we now proceed to the general definition, but we recommend the reader to always keep the above special case in mind (perhaps even with $A \in \mathcal{B}(\mathtt{X})$, so that $\mathscr{A}^t = e^{At}$).

We need some more definitions. Set $\mathrm{L}^2_\omega := \{e^{\omega\cdot} f \mid f \in \mathrm{L}^2\}$. The *Laplace transform* of $u : \mathbb{R}_+ \to \mathtt{U}$ is given by $\widehat{u}(s) := \int_0^\infty e^{-st} u(t)\,dt$. The map $u \mapsto \widehat{u}$ is an isomorphism of $\mathrm{L}^2_\omega(\mathbb{R}_+; \mathtt{U})$ onto $\mathcal{H}^2_\omega(\mathtt{U})$.

Given $P \in \mathcal{H}^\infty_\omega(\mathtt{U}, \mathtt{Y})$, we define its I/O map $\mathscr{D}_P : \mathrm{L}^2_\omega(\mathbb{R}_+; \mathtt{U}) \to \mathrm{L}^2_\omega(\mathbb{R}_+; \mathtt{Y})$ by $\widehat{\mathscr{D}_P u} := P\widehat{u}$; It follows that $\|\mathscr{D}_P\|_{\mathcal{B}} = \|P\|_{\mathcal{H}^\infty_\omega}$ [Wei91]. We identify $\mathscr{D}_P$ with its unique translation-invariant extension $\in \mathcal{B}(\mathrm{L}^2_\omega(\mathbb{R}; \mathtt{U}), \mathrm{L}^2_\omega(\mathbb{R}; \mathtt{Y}))$. (Translation-invariant means that $\mathscr{D}_P \tau^t = \tau^t \mathscr{D}_P\ \forall t \in \mathbb{R}$, where $(\tau^t u)(r) := u(t + r)$. This map satisfies $\widehat{\mathscr{D}_P u} = P\widehat{u}$ a.e. on $\omega + i\mathbb{R}$, where P refers to the boundary trace of P [Mik08a].) We set $\pi_+ u := \begin{cases} u(t), & t \ge 0; \\ 0, & t < 0 \end{cases}$, $\pi_- := I - \pi_+$.

Now we give an exact definition of WPLSs. Explanations will follow.

Definition 5.1 (WPLS) *Let* $\omega \in \mathbb{R}$. *An* ω-stable well-posed linear system on $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$ *is a quadruple* $\Sigma = \left[\begin{smallmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{smallmatrix}\right]$, *where* $\mathscr{A}^t$, $\mathscr{B}$, $\mathscr{C}$, *and* $\mathscr{D}$ *are bounded linear operators of the following type:*

1. $\mathscr{A}^\cdot$ *is a strongly continuous semigroup of bounded linear operators on* $\mathtt{X}$ *satisfying* $\sup_{t\ge0} \|e^{-\omega t}\mathscr{A}^t\|_{\mathtt{X}} < \infty$;

2. $\mathscr{B}\colon \mathrm{L}^2_\omega(\mathbb{R}; \mathtt{U}) \to \mathtt{X}$ *satisfies* $\mathscr{A}^t\mathscr{B}u = \mathscr{B}\tau^t\pi_- u$ *for all* $u \in \mathrm{L}^2_\omega(\mathbb{R}; \mathtt{U})$ *and* $t \in \mathbb{R}_+$;

3. $\mathscr{C}\colon \mathtt{X} \to \mathrm{L}^2_\omega(\mathbb{R}; \mathtt{Y})$ *satisfies* $\mathscr{C}\mathscr{A}^t x = \pi_+\tau^t\mathscr{C}x$ *for all* $x \in \mathtt{X}$ *and* $t \in \mathbb{R}_+$;

4. $\mathscr{D}\colon \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U}) \to \mathrm{L}^2_\omega(\mathbb{R};\mathtt{Y})$ *satisfies* $\tau^t\mathscr{D}u = \mathscr{D}\tau^t u$, $\pi_-\mathscr{D}\pi_+u = 0$, *and* $\pi_+\mathscr{D}\pi_-u = \mathscr{C}\mathscr{B}u$ *for all* $u \in \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U})$ *and* $t \in \mathbb{R}$.

Also the *dual* $\Sigma^{\mathrm{d}} := \left[\begin{smallmatrix}\mathscr{A}^* & \mathscr{C}^*\mathcal{R} \\ \mathcal{R}\mathscr{B}^* & \mathcal{R}\mathscr{D}^*\mathcal{R}\end{smallmatrix}\right]$ of Σ is an ω-stable WPLS, where the reflection $\mathcal{R}$ is defined through $(\mathcal{R}u)(t) := u(-t)$.

Let A be the generator of $\mathscr{A}$. There exist unique linear operators $C : \mathrm{Dom}(A) \to \mathtt{X}$ and $B : \mathtt{U} \to (\mathrm{Dom}(A^*))^*$ such that for every $t \geq 0$ we have $(\mathscr{C}x_0)(t) = C\mathscr{A}^t x_0$ ($x_0 \in \mathrm{Dom}(A)$) and $\mathscr{B}v := \int_0^\infty \mathscr{A}^r Bv(-r)\,dt$ (for compactly supported $v \in \mathrm{L}^2$).

The *transfer function* $\hat{\mathscr{D}}$ of Σ is the unique function $\hat{\mathscr{D}} \in \mathcal{H}^\infty_\infty(\mathtt{U},\mathtt{Y})$ for which $\widehat{\mathscr{D}u} = \hat{\mathscr{D}}\hat{u}$ for every $u \in \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{U})$. The system Σ is uniquely determined by A, B, C and $\hat{\mathscr{D}}(s)$ for a fixed s; also the converse is true. Therefore, we sometimes write $\Sigma = (A, B, C; \hat{\mathscr{D}})$.

We mention that Definition 5.1 is equivalent to the assumption that A, B, C are as above, $\hat{\mathscr{D}}(s) - \hat{\mathscr{D}}(z) = (s-z)C(s-A)^{-1}(z-A)^{-1}B$ is bounded on some right half-plane, and $\mathscr{C}$ and $\mathscr{B}^*$ extend to $\mathtt{X} \to \mathrm{L}^2_{\mathrm{loc}}$. An equivalent condition is that the map $x_0, u \mapsto x(t), y$ defined below is bounded $\mathtt{X} \times \mathrm{L}^2([0,T];\mathtt{U}) \to \mathtt{X} \times \mathrm{L}^2([0,T];\mathtt{Y})$ for some (hence every) $T > 0$.

For any *initial state* $x_0 \in \mathtt{X}$ and *input* $u \in \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{U})$ we denote the *state* by $x(t) := \mathscr{A}^t x_0 + \mathscr{B}u(t+\cdot)$, and the *output* by $y := \mathscr{C}x_0 + \mathscr{D}u \in \mathrm{L}^2_\omega$. There exists a "compatible extension" $\overline{C}$ of C [SW02] [Sta05] [Mik02] and $D \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ such that $\hat{\mathscr{D}}(s) = D + \overline{C}(s-A)^{-1}B$ for $\mathrm{Re}\, s > \omega$, and, in a certain sense, x and y form the unique solution of $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = \overline{C}x(t) + Du(t)$, $x(0) = x_0$. If B or C is *bounded* (i.e., $B \in \mathcal{B}(\mathtt{U},\mathtt{X})$ or $C \in \mathcal{B}(\mathtt{X},\mathtt{Y})$), then $\overline{C} = C$, the limit $\lim_{s\to\infty} \hat{\mathscr{D}}(s)$ exists and equals D, and Σ^{d} is determined by $\left(\begin{smallmatrix}A^* & C^* \\ B^* & D^*\end{smallmatrix}\right)$.

It is well known that, by using unique extensions/restrictions of $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$, an ω-stable WPLS Σ can be seen as a β-stable WPLS for every $\beta > \omega$.

The WPLS Σ is called *stable* if $y \in \mathrm{L}^2$ and x is bounded for every $x_0 \in \mathtt{X}$ and every $u \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{U})$ (or equivalently, Σ is a 0-stable system); *output-stable* if $y \in \mathrm{L}^2$ for every $x_0 \in \mathtt{X}$ when $u = 0$; *exponentially stable* if it is ω-stable for some $\omega < 0$ (or equivalently, 1. holds for some $\omega < 0$).

In the simplest case, a *state feedback* means the substitution $u(t) = Fx(t)$ for some $F \in \mathcal{B}(\mathtt{X},\mathtt{U})$. In general, (well-posed) state feedback may have F unbounded, contain a feedforward $G \in \mathcal{B}(\mathtt{U})$, and allow for an external input or disturbation, say $u_\circlearrowleft$, so the substitution becomes $u(t) = \overline{F}x(t) + Gu(t) + u_\circlearrowleft(t)$, i.e., $u = \mathscr{F}x_0 + \mathscr{G}u + u_\circlearrowleft = (I-\mathscr{G})^{-1}\mathscr{F}x_0 + (I-\mathscr{G})^{-1}\mathscr{G}u$, with the well-posedness conditions that $\left[\begin{smallmatrix}\mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \\ \mathscr{F} & \mathscr{G}\end{smallmatrix}\right]$ is a WPLS on $(\mathtt{U},\mathtt{X},\mathtt{Y}\times\mathtt{U})$ and $M := (I-\hat{\mathscr{G}})^{-1} \in \mathcal{H}^\infty_\infty$.

If $\hat{\mathscr{G}}(+\infty)$ exists (uniformly), then one can normalize the feedforward term G to zero without affecting the closed-loop maps $x_0 \mapsto (x, y, u)$. In any case, $\hat{\mathscr{D}} = NM^{-1}$ is a right factorization and $\left[\begin{smallmatrix}N \\ M\end{smallmatrix}\right]$ is the closed-loop I/O map $u_\circlearrowleft \mapsto \left[\begin{smallmatrix}y \\ u\end{smallmatrix}\right]$, where $N := \hat{\mathscr{D}}M$.

The WPLS Σ is called *stabilizable* (resp., *output-stabilizable*) if some state feedback makes the resulting (closed-loop) system (with input $u_\circlearrowleft$ and output $\left[\begin{smallmatrix}y \\ u\end{smallmatrix}\right]$) stable (resp., output-stable). Moreover, Σ is called *detectable* if its dual Σ^{d} is stabilizable. See [Sta05], [Sta98a], [Mik07d] or [Mik02] for *joint* stabilizability and detectability.

The *Finite Cost Condition* means that for every $x_0 \in \mathtt{X}$ there exists $u \in \mathrm{L}^2(\mathbb{R}_+; \mathtt{U})$ such that $y \in \mathrm{L}^2$. The Finite Cost Condition holds iff the system is output-stabilizable; a third equivalent condition is that there exists a state feedback (called the *LQ-optimal* state feedback) that minimizes the "LQR cost" $\int_0^\infty (\|y(t)\|_{\mathtt{Y}}^2 + \|u(t)\|_{\mathtt{U}}^2)\,dt$ for every $x_0 \in \mathtt{X}$ [Mik06b]. Moreover, using the LQ-optimal state-feedback, the above factorization $\hat{\mathscr{D}} = NM^{-1}$ becomes a w.r.c.f. [Mik06b], as in the finite-dimensional case. Analogous discrete-time results (and definitions) are given in [Mik07g].

A *realization* of P means a WPLS whose transfer function $\hat{\mathscr{D}}$ equals P.

Recall that Theorem 1.1 is contained in Theorem 3.2. An alternative proof of the first claim of Theorem 1.1 is given in the proof below. Conversely, one could use this connection (or the Cayley transform) to reduce several main results of [Mik06b] to [Mik07g].

Proof of Theorem 1.2: 1° The implications "(iv)⇒(iii)" and "(ii)⇒(i)" are trivial. The implications "(iii)⇔(v)⇒(ii)" are from [Mik06b, Corollaries 5.3 & 5.2]. The implication "(i)⇒(iv)" was established in [Mik02, Lemma 6.6.29]; the proof of [Sta05, Theorem 8.4.6(iii)] can alternatively used (or that of [Sta98a, Theorem 4.4] with additional modifications).

2° The "Moreover" paragraph is contained in [Mik06b, Lemma 4.4].

3° Let $P = NM^{-1}$ be a normalized w.r.c.f. Let Σ be an output-stabilizable realization of P. The corresponding LQ-optimal state-feedback yields a normalized w.r.c.f. $P = N_0 M_0^{-1}$, by [Mik06b, Lemma 4.4]. By [Mik06b, Lemma A.5], the nonuniqueness of the LQ-optimal state-feedback (which may include some feedthrough) allows us to replace N_0 and M_0 by $N_0 V$ and $M_0 V$ for any $V \in \mathcal{GB}(\mathtt{U})$. By Theorem 1.1, this way we obtain N and M. □

The stabilizable realization in [Mik02, Lemma 6.6.29] was constructed in terms of N and M. For an output-stabilizable realization in terms of P only, see Theorem 6.4. From [Mik02, Lemma 6.6.29] we see that the stabilizable realization in Theorem 1.2(iv) can be chosen so as to be "strongly stabilizable"; this means that the closed-loop system satisfies the additional constraint that $x(t) \to 0$ for every $x_0 \in \mathtt{X}$ (when $u_\circlearrowleft = 0$). The same applies to the realizations in Theorem 1.3(iv)&(v) [Mik02, Theorem 6.6.28].

By applying the Cayley transform $s \mapsto \frac{1-s}{1+s}$ to [Mik07g, Lemma 4.1] we observe that any r.c.f. can be extended to a d.c.f.:

Lemma 5.2 *If $N \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ and $M \in \mathcal{H}^\infty(\mathtt{U})$ are r.c., then there exist a closed subspace $\mathtt{Y}_2 \subset \mathtt{Y}$ and functions $X, Y \in \mathcal{H}^\infty$ such that $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] \in \mathcal{G}\mathcal{H}^\infty(\mathtt{Y}_2 \times \mathtt{U}, \mathtt{Y} \times \mathtt{U})$.*

If $\dim \mathtt{U} < \infty$ or $M(z) \in \mathcal{GB}(\mathtt{U})$ for some $z \in \mathbb{C}^+$, then we can choose X and Y above so that $\mathtt{Y}_2 = \mathtt{Y}$.

In particular, any r.c.f. can be extended to a d.c.f.

Lemma 5.3 *If a WPLS and its dual are output-stabilizable, then its transfer function has a d.c.f.*

Proof: (We write DT for discrete time; the system-theoretic details are given in [Mik07g].) Choose some $\alpha > \max\{0, \omega_A\}$. By Lemma A.1(b)&(d), the DT systems $\Sigma_{\rm d}$ and $\Sigma_{\rm d}^{\rm d}$ satisfy the DT Finite Cost Condition, hence $\widehat{\mathscr{D}_{\rm d}}$ has a DT-r.c.f. NM^{-1}, by [Mik07g, Lemma 4.2 & Proposition 3.1]. Redefine the functions by applying $\circ\psi_\alpha$, where $\psi_\alpha(s) := \frac{\alpha - s}{\bar{\alpha}+s}$, to obtain, by Lemma A.1(c), a α-r.c.f. NM^{-1} of the original transfer function $\hat{\mathscr{D}}$. By Theorem 3.1(a), NM^{-1} is a r.c.f.; by Lemma 5.2, $\hat{\mathscr{D}}$ has a d.c.f. □

Proof of Theorem 1.3: The implication "(iii)⇒(ii)" is Lemma 5.3, "(i)⇒(ii)" is from Lemma 5.2, "(ii)⇔(v)" is from [Sta98a, Theorem 4.4], and the implications "(v)⇒(iv)⇒(iii)" and "(ii)⇒(i)" are trivial. □

6 Existence of weakly coprime factorizations

In this section we present two more conditions that are equivalent to the existence of a w.r.c.f. One is that $P = NM^{-1}$, where N and M are "almost $\mathcal{H}^\infty$" (Theorem 6.2), the other that the generalized Hankel range of P is stabilized by the generalized Toeplitz range of P (Theorem 6.3).

We also present a general method for constructing an output-stabilizable realization in terms of the transfer function (instead of a factorization), in Theorem 6.4. If P has a r.c.f., then the algorithm in Remark 6.5 yields a r.c.f., a d.c.f., a robust stabilizing controller and a stabilizable and detectable realization of P, constructively.

By $\mathrm{L}^2_{\rm c}$ we denote the L^2 functions that have a compact support. If $\mathscr{D}_P$ (see Section 5) maps $\mathrm{L}^2_{\rm c} \to \mathrm{L}^2$, then P is almost $\mathcal{H}^\infty$:

Lemma 6.1 **($\mathscr{D}\mathrm{L}^2_{\rm c} \subset \mathrm{L}^2$)** *Let $P \in \mathcal{H}^\infty_\infty(\mathtt{U}, \mathtt{Y})$. Assume that $\mathscr{D}_P \mathrm{L}^2([0,T];\mathtt{U}) \subset \mathrm{L}^2(\mathbb{R};\mathtt{Y})$ for some $T > 0$.*

For any $\alpha, \omega \in \mathbb{R}$ such that $\alpha < 0 < \omega$, we have $P \in \mathcal{H}^\infty_\omega$, $\mathscr{D}_P \pi_+ : \mathrm{L}^2_\alpha \to \mathrm{L}^2$, and $\mathscr{D}_P \pi_- : \mathrm{L}^2_\omega \to \mathrm{L}^2$, continuously.

Lemma 6.1 follows from [Mik07g, Lemma 5.2] by time discretization (set $\mathtt{U}' := \mathrm{L}^2([0,T);\mathtt{U})$, $\mathtt{Y}' := \mathrm{L}^2([0,T);\mathtt{Y})$, $\tilde{u}_k := \pi_{[0,T)}\tau^{kT}u$, where $\pi_{[0,T)}$ is the projection to $[0,T)$, so that $\|\tilde{u}\|_{\ell^2} = \|u\|_2$ and $u \mapsto \tilde{u}$ is an isomorphism of $\mathrm{L}^2_\omega(\mathbb{R};\mathtt{U})$ onto $\ell^2_r(\mathbb{Z};\mathtt{U}')$, where $r := \mathrm{e}^\omega$, and $u \mapsto \tilde{u}$ commutes with $\pi_\pm$). Further properties of such $\mathscr{D}_P$ are given in [Mik02, Lemma 2.1.3].

The I/O map of any output-stable system maps $\mathrm{L}^2_{\rm c} \to \mathrm{L}^2$ [Mik02, Lemma 6.1.11] but not necessarily $\mathrm{L}^2 \to \mathrm{L}^2$ (which corresponds to $\mathcal{H}^\infty$ transfer functions), thus motivating Lemma 6.1. Next we give another motivation.

We show that condition (i) in Theorem 1.2 has a formally weaker equivalent form: if $\mathscr{D}_P$ has an "almost-right factorization" with $\mathscr{D}_N, \mathscr{D}_M$ mapping $\mathrm{L}^2_{\rm c} \to \mathrm{L}^2$, then $\mathscr{D}_P$ has a right factorization:

Theorem 6.2 *If $P = NM^{-1}$, where $N \in \mathcal{H}^\infty_\infty(\mathtt{U}, \mathtt{Y})$, $M \in \mathcal{GH}^\infty_\infty(\mathtt{U})$ and $\begin{bmatrix} \mathscr{D}_N \\ \mathscr{D}_M \end{bmatrix} \mathrm{L}^2_{\rm c} \subset \mathrm{L}^2$, then P has a w.r.c.f.*

(By time-invariance, $\mathscr{D}_N \mathrm{L}^2_{\mathrm{c}} \subset \mathrm{L}^2$ iff $\mathscr{D}_N \mathrm{L}^2([0,T];\mathtt{U}) \subset \mathrm{L}^2(\mathbb{R};\mathtt{Y})$ for some $T > 0$.)

Proof: A map of the type in Lemma 6.1 has the output-stable (shift-semigroup) realization $\left[\begin{array}{c|c} \tau & \pi_- \\ \hline \pi_+ \mathscr{D}\pi_- & \mathscr{D} \end{array}\right]$ with state space $\mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U})$. Apply this to $\left[\begin{smallmatrix} \mathscr{D}_N \\ \mathscr{D}_M \end{smallmatrix}\right]$, feed the second output times $-I$ back to the input and remove the bottom row to obtain an output-stabilizable realization of $P = NM^{-1}$ (this part is written out in the proof of [Mik02, Lemma 6.6.29], or less directly in that of [Sta05, Theorem 8.4.6(iii)]; in discrete-time this is written out in part 6° of the proof of Theorem 8.4). The conclusion then follows from Theorem 1.2. □

Two more conditions (the existence of a "Nev/Nev" or of a "$\mathcal{H}^2_{\mathrm{strong}}/\mathcal{H}^2_{\mathrm{strong}}$" factorization) equivalent to Theorem 1.2(i) are listed below Theorem 9.2. Still another is that the "generalized Hankel range" $\pi_+\mathscr{D}_P\pi_-$ lies in the "generalized Toeplitz range" $\pi_+\mathscr{D}_P\pi_+$ plus $\mathrm{L}^2(\mathbb{R}_+;\mathtt{U})$, i.e., that "the Toeplitz range of P stabilizes the Hankel range of P". Below we formulate that condition in four equivalent ways:

Theorem 6.3 *Let $P \in \mathcal{H}^\infty_\gamma(\mathtt{U},\mathtt{Y})$ for some $\gamma \geq 0$. Then also each of the following conditions are equivalent to the existence of a right factorization of P:*

(vi) *There exists $\omega \geq \gamma$ such that $\pi_+\mathscr{D}_P\pi_-[\mathrm{L}^2_\omega] \subset \pi_+\mathscr{D}_P\pi_+[\mathrm{L}^2] + \mathrm{L}^2(\mathbb{R}_+;\mathtt{Y})$.*

(vi') *There exists $\omega \geq \gamma$ such that for any $v \in \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U})$ there exists $u \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{U})$ such that $\mathscr{D}_P(v+u) \in \mathrm{L}^2$.*

(vi") *There exists $\omega \geq \gamma$ such that for any $v \in \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U})$ there exists $u \in \mathrm{L}^2(\mathbb{R};\mathtt{U})$ such that $\pi_+\mathscr{D}_P(v+u) \in \mathrm{L}^2$.*

(vi"') *There exists $\omega \geq \gamma$ such that for any $v \in \mathcal{H}^2(\mathbb{C}^-_\omega;\mathtt{U})$ there exists $f \in \mathcal{H}^2(\mathtt{U})$ such that $\widehat{\pi}_+P(v+f) \in \mathcal{H}^2$.*

Proof: 1° *(vi)–(vi"'),(i):* Now $\pi_-\mathscr{D}_P(v+u) = \pi_-\mathscr{D}_P v \in \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{Y}) \subset \mathrm{L}^2$ (because $\omega \geq 0$), hence one easily observes that (vi)–(vi"') are equivalent (even with the same ω). By Theorem 6.4 below, (vi") implies Theorem 1.2(iii) (which is equivalent to (i)).

2° *(i)⇒(vi'):* Take $\omega \geq 0$ such that $M \in \mathcal{G}\mathcal{H}^\infty_\omega$. Let $v \in \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U})$ be arbitrary. Then $\tilde{v} := \pi_-\mathscr{D}_M^{-1}v \in \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U}) \subset \mathrm{L}^2(\mathbb{R}_-;\mathtt{U})$, hence $\tilde{u},\tilde{y} \in \mathrm{L}^2$, where $\tilde{u} := \pi_+\mathscr{D}_M\tilde{v}$, $\tilde{y} := \pi_+\mathscr{D}_N\tilde{v}$. But $v = \pi_-\mathscr{D}_M\mathscr{D}_M^{-1}v = \pi_-\mathscr{D}_M\tilde{v}$, hence

$$\pi_+\mathscr{D}_P(v+\tilde{u}) = \pi_+\mathscr{D}_P(\pi_-\mathscr{D}_M\tilde{v} + \pi_+\mathscr{D}_M\tilde{v}) = \pi_+\mathscr{D}_P\mathscr{D}_M\tilde{v} = \tilde{y} \in \mathrm{L}^2. \quad (4)$$

Therefore, (vi') holds (set $u := \tilde{u}$). □

Further conditions equivalent to those in Theorems Theorem 1.2 and 1.3 are given in [Mik07d] and [Mik07c].

In the proof of Theorem 1.2, a stabilizable realization of P was constructed using a right factorization of P. Even if no such factorization is given, we can use the following formula to obtain an output-stabilizable realization (namely the standard shift-semigroup realization [Sal89] [Sta98a]).

Theorem 6.4 *If the condition (vi") in Theorem 6.3 holds, then the WPLS* $\left[\begin{array}{c|c} \tau & \pi_- \\ \hline \pi_+ \mathscr{D}_P \pi_- & \mathscr{D}_P \end{array}\right]$ *with the state space* $\mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U})$ *is output-stabilizable.*

Indeed, condition (vi") is the Finite Cost Condition for that system, and Finite Cost Condition is equivalent to output-stabilizability (Section 5). See Theorem 1.2 for the converse.

For any function P having a right factorization (resp. a r.c.f.), we remark below constructive formulae for 1. a stabilizable realization, 2. normalized w.r.c.f. (resp. 3. normalized r.c.f., 4. stabilizable and detectable realization, 5. d.c.f. and robust stabilizing controllers). In 2., 3. and 5., one can also start from any fixed realization that satisfies certain weak stabilizability conditions.

Remarks 6.5 (Constructive formulae)

1. **(Stabilizable realization)** If a function P has a right factorization, then Theorem 6.4 provides the formula for an output-stabilizable realization of P, and the proof of Theorem 1.2 provides the formula for a stabilizable realization of P.

2. **(Normalized w.r.c.f.)** Given an output-stabilizable realization of P, a normalized w.r.c.f. of P is constructed in Theorem 1.2 (see its footnote and [Mik06b] for the complex general formulae, which can be simplified in numerous special cases found in the literature including the references of [Mik06b] and [Mik07a]).

3. **(Normalized r.c.f.)** If P has a r.c.f., then 1.–2. provide a normalized r.c.f. of P, by Theorem 1.1.

4. **(Stabilizable and detectable realization)** If (and only if) P has a r.c.f., then also the dual of the stabilizable realization mentioned in 1. is stabilizable (this will be shown in [Mik07d]).

5. **(D.c.f. and robust controllers)** Assume that a system $\Sigma := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ and its dual are output-stabilizable (e.g., that Σ is the realization constructed in 4. above). Then constructive formulae for a d.c.f. (and its inverse) of the transfer function of Σ are provided in [CO06, Theorem 8.9 & Remark 8.10] under the assumption that $0 \in \rho(A)$ (or $\rho(A) \cap i\mathbb{R} \neq \emptyset$ with translation; otherwise one must, e.g., use the Cayley transform and corresponding discrete-time results [CO08] [Mik07g]).

Finally, constructive formulae for *robust* stabilizing controllers for the transfer function of Σ can be found in [Cur06, Theorem 22] (assuming that $0 \in \rho(A)$ or as above). $\triangleleft$

In [Mik07f] the author shall show that if, in 1. above, we start with a function P that is *real-symmetric*, i.e., whose Fourier coefficients are real, then we obtain a real realization (i.e., A, B and C are real and P is real-symmetric; if $D := P(+\infty)$ exists, then it is real too). If we start with a real system in 2., then, in 2.–5., all operators are real and all functions are real-symmetric. Thus, the numerous control and factorization problems associated with Remark 6.5 and with Theorems 1.1–1.3 all have real/real-symmetric solutions (if they have any solutions at all) provided that the original data is real.

7 Further results

In this section we present further results on left invertibility and coprime factorization, both standard (Bézout), weak and α-weak: the properties of their reciprocals and Hankel operators, and the relations between different values of α. These results will be applied in the study of dynamic stabilization and state-feedback stabilization in later articles.

Our first lemma sheds extra light to how coprime factorizations behave in the operator-valued case and thus also to the question, what kind of stabilizing compensators are equivalent.

If NM^{-1} has an α-r.c.f. and a β-r.c.f., then these are the same (modulo a unit):

Lemma 7.1 (α-r.c.f.= β-r.c.f.) *Let NM^{-1} be a $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued α-right factorization, $\alpha,\beta\in\mathbb{C}^+$, $M(\beta)\in\mathcal{GB}$, and let $N_1M_1^{-1}$ be an α-r.c.f. of NM^{-1}.*

If $N_2M_2^{-1}$ is a β-w.r.c.f. of NM^{-1}, then $\left[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}\right]=\left[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}\right]T$ for some $T\in\mathcal{H}^\infty(\mathtt{U})$, and $M_2(\alpha),T(\alpha)\in\mathcal{GB}$. If $N_2M_2^{-1}$ is a β-r.c.f. of NM^{-1}, then $\left[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}\right]=\left[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}\right]T$ for some $T\in\mathcal{G}\mathcal{H}^\infty(\mathtt{U})$.

(If $\dim\mathtt{U}<\infty$, then this holds with "w.r.c.f." in place of "r.c.f.", and then $N_1M_1^{-1}$ is a β-w.r.c.f. too, by Theorems 2.8 and 2.1, but we do not know if the same holds in general.)

Proof of Lemma 7.1: Let $N_2M_2^{-1}$ be a β-w.r.c.f. of NM^{-1} (Theorem 1.1). Then

$$\begin{bmatrix}N\\M\end{bmatrix}=\begin{bmatrix}N_1\\M_1\end{bmatrix}V_1=\begin{bmatrix}N_2\\M_2\end{bmatrix}V_2 \tag{5}$$

for some $V_k\in\mathcal{H}^\infty(\mathtt{U})$ such that $V_1(\alpha),V_2(\beta)\in\mathcal{GB}(\mathtt{U})$, by Theorem 1.1. Therefore, $V_1=TV_2$, where $T:=\begin{bmatrix}-\tilde{Y}_1 & \tilde{X}_1\end{bmatrix}\left[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}\right]\in\mathcal{H}^\infty(\mathtt{U})$ (fix some $\begin{bmatrix}-\tilde{Y}_1 & \tilde{X}_1\end{bmatrix}\in\mathcal{H}^\infty$ such that $\begin{bmatrix}-\tilde{Y}_1 & \tilde{X}_1\end{bmatrix}\left[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}\right]=I$). In particular, $\left[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}\right]T=\left[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}\right]$ near β (by (5)), hence on $\mathbb{C}^+$.

Since $\left[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}\right](z)$ is one-to-one for each $z\in\mathbb{C}^+$, by Theorem 2.2, so is T too, hence $M_2(\alpha)$ is one-to-one (because $M_1T=M_2$). But $M_2(\alpha)V_2(\alpha)=M(\alpha)\in\mathcal{GB}(\mathtt{U})$, hence $M_2(\alpha)$ is onto, hence $M_2(\alpha),T(\alpha)\in\mathcal{GB}(\mathtt{U})$.

If N_2 and M_2 are r.c., then they are α-w.r.c. and hence $T\in\mathcal{G}\mathcal{H}^\infty$, by Theorem 2.1. □

If the "F_wF_r-factorizations" (see Theorem 2.10(a)) of F at two different points of $\mathbb{C}^+$ both have a left-invertible F_w, then the factorizations are essentially equal:

Lemma 7.2 *Let $F_1V_1=F_2V_2$, $G_kF_k=I$, where $F_k,V_k,G_k\in\mathcal{H}^\infty$, $V_k(\alpha_k)\in\mathcal{GB}(\mathtt{U})$, $\alpha_k\in\mathbb{C}^+$ ($k=1,2$). Then $F_2=F_1T$ for some $T\in\mathcal{G}\mathcal{H}^\infty(\mathtt{U})$.*

Proof: Now $V_1=G_1F_2V_2$ and $V_2=G_2F_1V_1=G_2F_1G_1F_2V_2$, hence $G_2F_1G_1F_2=I$ near α_2, hence on $\mathbb{C}^+$. Similarly, $TS=I$, where $T:=G_1F_2$, $S:=G_2F_1$. But $F_2V_2=F_1V_1=F_1TV_2$, hence $F_2=F_1T$ near α_2, hence on $\mathbb{C}^+$. □

The "reciprocals" [OC04] of [w.]r.c. maps are [w.]r.c.:

Lemma 7.3 *Let $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$. Set $F_r := F(1/\cdot)$. Then F is left-invertible iff F_r is left-invertible. Moreover, F is α-weakly left-invertible iff F_r is $1/\alpha$-weakly left-invertible. If $F^*F \geq \epsilon I$ and $F_r^*F_r \geq \epsilon I > 0$ on some right half-plane, then F is weakly left-invertible iff F_r is weakly left-invertible.*

The above coercivity assumptions are not superfluous: the w.r.c.f. NM^{-1} of Example 4.3 does not even have $N_r := N(1/\cdot)$ and $M_r := (1/\cdot)$ w.r.c., by Example 4.2. By the above, this could not happen if M_r^{-1} was proper.
Proof: The first claim is obvious (if $GF = I$, then $G_rF_r = I$). Set $(\widehat{\mathcal{T}}f)(s) := \frac{1}{s}f(\frac{1}{s})$. One easily verifies that $\widehat{\mathcal{T}}$ is unitary on $\mathcal{H}^2(\mathtt{U})$, that $\widehat{\mathcal{T}}f \in \mathcal{H}^2 \Leftrightarrow f \in \mathcal{H}^2$, and that a function f is α-proper iff $\widehat{\mathcal{T}}f$ is $1/\alpha$-proper. Now $Ff \in \mathcal{H}^2 \Leftrightarrow \mathcal{H}^2 \ni \frac{1}{\cdot}(Ff)(\frac{1}{\cdot}) = F_r\widehat{\mathcal{T}}f$, hence the second claim follows easily. Theorem 3.3 now provides the last claim. □

We establish the fact that $\operatorname{Ker}(\Gamma_P) = \operatorname{Ker}(\Gamma_{M^{-1}})$, where $\Gamma_P := \pi_+\mathscr{D}_P\pi_-$ stands for the time-domain Hankel operator of P.

Lemma 7.4 **($\Gamma_{\mathscr{D}} \sim \Gamma_{\mathscr{M}^{-1}}$)** *Let $P = NM^{-1}$ be a normalized w.r.c.f., where $M \in \mathcal{GH}^\infty_\omega(\mathtt{U})$ for some $\omega \geq 0$. Set $\mathscr{D} := \mathscr{D}_P$, $\mathscr{M} := \mathscr{D}_M$, $\mathscr{N} := \mathscr{D}_N$. Then the following hold for every $u \in \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U})$.*

- **(a)** $\mathscr{D}u \in \mathrm{L}^2 \Leftrightarrow \mathscr{M}^{-1}u \in \mathrm{L}^2 \Leftrightarrow \pi_+\mathscr{D}u \in \mathrm{L}^2 \Leftrightarrow \pi_+\mathscr{M}^{-1}u \in \mathrm{L}^2$.
- **(b)** $\pi_+\mathscr{D}u = 0 \Leftrightarrow \pi_+\mathscr{M}^{-1}u = 0$.

Moreover, part (a) above holds even without the normalization assumption (that $\mathscr{N}^*\mathscr{N} + \mathscr{M}^*\mathscr{M} = I$).

(Part (b) is not true for non-normalized [w.]r.c.f.'s; to observe this, take, e.g., $\mathscr{D} = 0 = \mathscr{N}$, $M(s) = (s+1)/(s+2) \in \mathcal{GH}^\infty$, $\omega = 0$.)
Proof: (a) Since π_- is continuous $\mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U}) \to \mathrm{L}^2$ and $\mathscr{D}, \mathscr{M}^{-1}$ are continuous $\mathrm{L}^2_\omega \to \mathrm{L}^2_\omega$, we have $\mathscr{D}u \in \mathrm{L}^2 \Leftrightarrow \pi_+\mathscr{D}u \in \mathrm{L}^2$, as well as $\mathscr{M}^{-1}u \in \mathrm{L}^2 \Leftrightarrow \pi_+\mathscr{M}^{-1}u \in \mathrm{L}^2$. If $\mathscr{M}^{-1}u \in \mathrm{L}^2$, then $\mathscr{D} = \mathscr{N}\mathscr{M}^{-1}u \in \mathrm{L}^2$. Finally, set $f := \pi_-\mathscr{M}^{-1}u$, $g := \pi_+\mathscr{M}^{-1}u$ and assume that $\mathscr{D}u \in \mathrm{L}^2$, i.e., $\mathrm{L}^2 \ni \left[\begin{smallmatrix}\mathscr{D}\\ I\end{smallmatrix}\right]u = \left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]\mathscr{M}^{-1}u = \left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right](f+g)$. Then $\mathrm{L}^2 \ni \left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]g$, because $f \in \pi_-\mathrm{L}^2_\omega \subset \mathrm{L}^2$, hence $g \in \mathrm{L}^2$ (because $\mathscr{N}$ and $\mathscr{M}$ are w.r.c.).
(b) Let f, g be as above. Then $\pi_+\mathscr{D}u = 0 \Leftrightarrow \pi_+\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right](f+g) = 0 \Leftrightarrow \|f+g\| = \|\pi_-\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right](f+g)\| = \|\pi_-\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]f\| \leq \|f\| \Leftrightarrow g = 0$ (because $f \perp g$). □

In [Mik07d] the following lemma will be used to establish the existence of a minimal realization whose LQ-optimal state feedback determines a predefined w.r.c.f.:

Lemma 7.5 *Let $\mathscr{D} = \mathscr{N}\mathscr{M}^{-1}$ be a normalized w.r.c.f. Then $\begin{bmatrix} I & -\mathscr{D}\end{bmatrix}$ is injective on $\overline{\operatorname{Ran}(\pi_+\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]\pi_-)}$; in particular, we have*
$\operatorname{Ker}(\begin{bmatrix} I & -\mathscr{D}\end{bmatrix}\pi_+\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]\pi_-) = \operatorname{Ker}(\pi_+\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]\pi_-)$ *(on $\mathrm{L}^2(\mathbb{R};\mathtt{U})$).*

(Take, e.g., $\mathscr{D} = 0 = \mathscr{N}$, $\hat{\mathscr{M}} = (s+1)/(s+2) \in \mathcal{G}\mathcal{H}^\infty$ to observe that neither claim is true for non-normalized r.c.f.'s.)

Proof: We prove the first claim; the second follows. Let $v_n \in \mathrm{L}^2(\mathbb{R}_-;\mathtt{U})$ $(n \in \mathbb{N})$ be such that $\left[\begin{smallmatrix} y\\ u\end{smallmatrix}\right] := \lim_n \pi_+ \left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]\pi_- v_n \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{U})$ exists, and assume that $\begin{bmatrix} I & -\mathscr{D}\end{bmatrix}\left[\begin{smallmatrix} y\\ u\end{smallmatrix}\right] = 0$, i.e., that $y = \mathscr{D}u$. Then $u_\circlearrowleft := \mathscr{M}^{-1}u \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{U})$, because $\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right] u_\circlearrowleft = \left[\begin{smallmatrix} y\\ u\end{smallmatrix}\right] \in \mathrm{L}^2$.

Now $\epsilon_n := \|\pi_+ \left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right](v_n - u_\circlearrowleft)\|_2 = \|\pi_+ \left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]\pi_- v_n - \left[\begin{smallmatrix} y\\ u\end{smallmatrix}\right]\|_2 \to 0$, as $n \to +\infty$. But $\|v_n\|_2^2 + \|u_\circlearrowleft\|_2^2 = \|v_n - u_\circlearrowleft\|_2^2 = \|\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right](v_n - u_\circlearrowleft)\|_2^2 = \|\pi_- \left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right](v_n - u_\circlearrowleft)\|_2^2 + \epsilon_n^2 = \|\pi_- \left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right] v_n\|_2^2 + \epsilon_n^2 \le \|v_n\|_2^2 + \epsilon_n^2$ $(n \in \mathbb{N})$, hence $\|u_\circlearrowleft\|_2^2 \le \epsilon_n^2$ $(n \in \mathbb{N})$. Therefore, $u_\circlearrowleft = 0$, hence $\left[\begin{smallmatrix} y\\ u\end{smallmatrix}\right] = 0$. Thus, $\begin{bmatrix} I & -\mathscr{D}\end{bmatrix}$ is injective. □

From the original proofs, mutatis mutandis (often alternatively through the Cayley transform), we observe the following.

Remark 7.6 (discrete-time setting) *Lemmata 7.1 and 7.2 and all results in Section 2 hold also in their* discrete-time forms*, by which we mean the following.*

Let $\alpha \in \mathbb{D}$. Replace everywhere the half-plane $\mathbb{C}^+$ by $\mathbb{D}$. Moreover, by $\mathcal{H}^2(\mathtt{U})$ we denote (the Hilbert space of) those holomorphic $f : \mathbb{D} \to \mathtt{U}$ for which $\|f\|_{\mathcal{H}^2} := \sup_{0<r<1} \|f(r\mathrm{e}^{i\cdot})\|_{\mathrm{L}^2} < \infty$.

Moreover, Lemmata 7.4 and 7.5 hold in the discrete-time setting described in [Mik07g]; e.g., replace $\mathrm{L}^2(\mathbb{R})$ *by* $\ell^2(\mathbb{Z})$ *and* $\mathrm{L}^2_\omega = \mathrm{e}^{\omega\cdot}\mathrm{L}^2$ *by* $\ell^2_r := r^{\cdot}\ell^2$*, where* $r > 0$*.*

Also Theorem 7.7 and Section 8 are new for discrete-time systems. Moreover, the discrete-time version of Theorem 6.2 holds; in fact, it is contained in the $\mathcal{H}^2_{\text{strong}}$ case of Theorem 8.4 (because "$\mathscr{D}_N$" maps finitely-nonzero sequences to ℓ^2 iff $N \in \mathcal{H}^2_{\text{strong}}(\mathbb{D};\mathcal{B}(\mathtt{U},\mathtt{Y}))$). Furthermore, in many such results, the disc $\mathbb{D}$ could be replaced by some other (simply) connected open subset of $\mathbb{C}$.

Many of our results that hold in the matrix-valued case do not hold in the general case. E.g., a function $g \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ may be injective everywhere but coercive only on a part of $\mathbb{C}^+$ (Example 4.6), or uniformly coercive on $\mathbb{C}^+$ but not left-invertible [Tre89]. Many such problems disappear if $g = G + fB$, where G and B are constants and B is compact. E.g., we can replace $\mathbb{C}^n$ by $\mathtt{U}$ in Theorem 2.20 if M is of such form (with f holomorphic on $\mathbb{C}^+$). An exponentially stable linear system having a compact input operator does have such a transfer function, so this is particularly relevant for exponentially stabilizable systems. (Actually, it suffices that the semigroup resolvent times the input operator is compact.)

We present below the corresponding generalization of Theorem 2.7. To simplify the notation, we work on the unit disc $\mathbb{D}$ (which corresponds to $\mathbb{C}^+$ and $\overline{\mathbb{D}}$ corresponds to $\overline{\mathbb{C}^+} \cup \{\infty\}$ through the Cayley Transform $s \mapsto (1-s)/(1+s)$).

Theorem 7.7 (left-invertible⇔no zeros) *Let $f : \mathbb{D} \to \mathcal{B}(\mathtt{X},\mathtt{Y})$ be holomorphic and bounded with a continuous extension to some closed $K \subset \overline{\mathbb{D}}$.*

Let $B \in \mathcal{B}(\mathtt{U}, \mathtt{X})$ be compact and let $G \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$. Set $g := G + fB$. Assume that $g^*g \geq \epsilon' I$ on $\mathbb{D} \setminus K$ for some $\epsilon' > 0$.

Then conditions (i) and (ii) below are equivalent. If B is a Hilbert–Schmidt operator or if g has a continuous extension to $\overline{\mathbb{D}}$, then (i)–(iii) are equivalent.

(i) *$g(z)u_0 \neq 0$ for all $z \in K$, $u_0 \in \mathtt{U} \setminus \{0\}$, and $g(z_0)$ is coercive for some $z_0 \in \mathbb{D}$;*

(ii) *$g(z)^*g(z) \geq \epsilon I$ for all $z \in \mathbb{D}$;*

(iii) *$hg = I$ for some bounded and holomorphic $h : \mathbb{D} \to \mathcal{B}(\mathtt{Y}, \mathtt{U})$.*

The equivalence of (i) and (ii) generalizes the equivalence of (ii)–(iv) of Theorem 2.7 (also with weak left-invertibility in place of α-weak left-invertibility).

By Example 4.6, not even the case $K = \overline{\mathbb{D}}$ of Theorem 7.7 would hold without the compactness assumption on B.

Proof of Theorem 7.7: 1° *(i)⇔(ii):* Since (ii) obviously implies (i), we assume (i). We also assume that G is coercive (replace G by $g(z_0)$ and f by $f - f(z_0)$).

To obtain a contradiction, assume that there does not exist $\epsilon > 0$ such that $\|g(z)u_0\| \geq \epsilon^{1/2}\|u_0\|$ for each $z \in \mathbb{D}$, $u_0 \neq 0$. Then there are $z_k \in K$, $u_k \in \mathtt{U}$ such that $\|u_k\| = 1$ $\forall k \in \mathbb{N}$ but $g(z_k)u_k \to 0$, as $k \to +\infty$. Taking subsequences if necessary, we have $z_k \to z$ and $Bu_k \to x$ for some $z \in K$, $x \in \mathtt{X}$. Thus, $Gu_k + f(z)x \to 0$, i.e., $Gu_k \to -f(z)x \in \mathtt{Y}$. Since G is coercive, we have $u_k \to u$ for some $u \in \mathtt{U}$. But $g(z)u = \lim_{k\to+\infty} g(z_k)u_k = 0$, hence $u = 0$, a contradiction (because $\|u\| = \lim_k \|u_k\| = 1$).

2° *(ii)⇔(iii):* This follows from [Tre04, Theorem 1.1(4)] if B is Hilbert–Schmidt and from [Vit03, p. 183] if g is continuous on $\overline{\mathbb{D}}$ (then so is some h satisfying (iii)). □

Using the above theorem, one can easily show that if a discrete-time system [Mik07g] has a compact input operator and its LQ-optimal control is power-stabilizing, then the corresponding w.r.c.f. is a r.c.f. (or equivalently, then the system is exponentially detectable). Also the continuous-time analogy of this claim holds and the assumption can be weakened [Mik07d].

The original proof of Theorem 2.20 shows that the zeros of M must be singularities of NM^{-1} (even if $\dim \mathtt{U} = \infty$). If M is of the form studied above, then any kind of singularities of M^{-1} are singularities of NM^{-1}.

Lemma 7.8 *Let NM^{-1} be a w.r.c.f., and $M = G + fB$, where $B : \mathtt{U} \to \mathtt{X}$ is compact, $G \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$, and $f : \mathbb{C}^+ \to \mathcal{B}(\mathtt{X}, \mathtt{Y})$ is holomorphic.*

Let $\Omega \subset \mathbb{C}^+$ be open and connected and contain a right half-plane. Then NM^{-1} has a holomorphic extension to Ω iff M^{-1} has, i.e., iff M is invertible on Ω.

The same obviously holds for α-w.r.c.f.'s too (where Ω should contain α instead).

Proof: Obviously, it suffices to assume that $P := NM^{-1}$ has a holomorphic extension to Ω and to prove that then M is invertible on Ω. For this end, it suffices to assume that $z_k \to z_0 \in \Omega$, $M(z_k) \in \mathcal{GB}$, as $k \to +\infty$, and $M(z_0) \notin \mathcal{GB}$, and to derive a contradiction. (Recall that $\{z \in \Omega \,|\, M(z) \in \mathcal{GB}\}$ is open.)

By Theorems 2.21 and 3.1(a), we can assume that $M(z_0)$ is not coercive, i.e., that $M(z_0)u_k \to 0$, as $k \to +\infty$, where $\|u_k\|_{\mathtt{U}} = 1$ $\forall k$. Replace G by $M(z_1)$ and f by $f - f(z_1)$ to have $G \in \mathcal{GB}(\mathtt{U})$. Pick a subsequence if necessary to have $Bu_k \to x_0$ for some $x_0 \in \mathtt{U}$. Then $Gu_k = M(z_0)u_k - f(z_0)Bu_k \to -f(z_0)x_0$, hence $u_k \to -G^{-1}f(z_0)x_0 =: u_0$, as $k \to +\infty$, where $\|u_0\| = 1$. Therefore, $M(z_0)u_0 = \lim_k M(z_k)u_k = 0$.

Now choose $g \in \mathcal{H}^2(\mathbb{C})$ such that $g(z_0) \neq 0$. Set $h := Mhu_0 \in D_P$ (see Theorem 2.19). Then $h(z_0) = g(z_0)M(z_0)u_0 = 0$ but yet $(Ph)(z_0) = g(z_0)N(z_0)u_0 \neq 0$, by Theorem 2.2, a contradiction, as required. □

8 Discrete-time w.r.c.f.'s (on $\mathbb{D}$)

In this section we show that also $\mathcal{H}^p_{\text{strong}}$ and Nevanlinna fractions and functions can be factorized as in Theorems 1.1 and 2.10. We define those classes below. Also Theorem 2.14 will be extended to $\mathcal{H}^p$ and to $\mathcal{H}^p_{\text{strong}}$, in Theorem 8.5. The implication (2) will be extended also to the Nevanlinna classes in place of $\mathcal{H}^2$, in Theorem 8.3. We prove all this in discrete-time terminology; analogous continuous-time results will be given in Theorem 9.2.

Thus, in this section we use different terminology than in the other sections. Throughout this section we assume that $1 \leq p \leq \infty$ and $\alpha \in \mathbb{D}$ (the unit disc), and, in all above definitions (e.g., of $\mathcal{H}^\infty$, and α-right factorizations) we replace the half-plane $\mathbb{C}^+$ by $\mathbb{D}$. Moreover, by $\mathcal{H}^p(\mathtt{B})$ we denote (the Banach space of) those holomorphic $f : \mathbb{D} \to \mathtt{B}$ for which $\|f\|_{\mathcal{H}^p} := \sup_{0<r<1} \|f(re^{i\cdot})\|_{\mathrm{L}^p} < \infty$,[3] when $\mathtt{B}$ is a complex Banach space.

Also in this *discrete-time form* all the results in Section 2 hold (use either the original proofs, mutatis mutandis, or Lemma 9.1). Therefore, we refer to them freely here too. The motivation to this terminology is that the transfer functions of all (resp., stable) discrete-time systems are 0-proper (resp., in $\mathcal{H}^\infty$); also the converses hold [Mik07g].

Bounded multiplication operators on $\mathcal{H}^p$ are $\mathcal{H}^\infty$ functions:

Lemma 8.1 *If V is an α-proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function,* $1 \leq p \leq \infty$ *and* $Vf \in \mathcal{H}^p(\mathtt{U})$ *for each* $f \in \mathcal{H}^p(\mathtt{X})$*, then* $V \in \mathcal{H}^\infty(\mathtt{X},\mathtt{U})$*.*

(The proof of [Mik07g, Lemma 3.5] yields this.)

[3] Thus, when $p = \infty$, the space $\mathcal{H}^p(\mathtt{U})$ is isometrically isomorphic to $\mathcal{H}^\infty(\mathbb{C},\mathtt{U})$, not to $\mathcal{H}^\infty(\mathtt{U}) := \mathcal{H}^\infty(\mathtt{U},\mathtt{U})$. This abuse does not cause ambiguity, because we use no other exponents than 2, p and ∞.

We write $f \in \mathrm{Nev}(\mathtt{U},\mathtt{Y})$ iff $f = \phi^{-1}g$, where $g \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$, $\phi \in \mathcal{H}^\infty(\mathbb{C})$, $\phi \neq 0$ on $\mathbb{D}$. Moreover, $f \in \mathrm{Nev}_+(\mathtt{U},\mathtt{Y})$ iff ϕ can be taken outer. One can show that $\mathrm{Nev}(\mathtt{U},\mathtt{Y})$ consists of those holomorphic $\phi^{-1}g$ for which $g \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ and $0 \not\equiv \phi \in \mathcal{H}^\infty(\mathbb{C})$, or equivalently, of those holomorphic $f : \mathbb{D} \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ for which $\sup_{0<r<1} \log^+ \|f(re^{i\cdot})\|_1 < \infty$ [RR85, p. 77].

By $\mathrm{Nev}_{\mathrm{strong}}(\mathtt{U},\mathtt{Y})$ we denote the set of (necessarily holomorphic) functions $F : \mathbb{D} \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ such that $Fu \in \mathrm{Nev}(\mathbb{C},\mathtt{Y})$ for every $u \in \mathtt{U}$. Similarly, $F \in \mathrm{Nev}_{+,\mathrm{strong}}(\mathtt{U},\mathtt{Y})$ iff $Fu \in \mathrm{Nev}_+(\mathbb{C},\mathtt{Y})$ $\forall u \in \mathtt{U}$. Moreover, $F \in \mathcal{H}^p_{\mathrm{strong}}$ if $Fu \in \mathcal{H}^p(\mathtt{Y})$ $\forall u \in \mathtt{U}$. We set $\mathrm{Nev}(\mathtt{U}) := \mathrm{Nev}(\mathtt{U},\mathtt{U})$; similarly for Nev_+, $\mathrm{Nev}_{\mathrm{strong}}$, $\mathrm{Nev}_{+,\mathrm{strong}}$ and $\mathcal{H}^p_{\mathrm{strong}}$.

We have the inclusions $\mathcal{H}^p(\mathcal{B}(\mathtt{U},\mathtt{Y})) \subset \mathrm{Nev}_+(\mathtt{U},\mathtt{Y}) \subset \mathrm{Nev}(\mathtt{U},\mathtt{Y})$ and hence also $\mathcal{H}^p_{\mathrm{strong}}(\mathtt{U},\mathtt{Y}) \subset \mathrm{Nev}_{+,\mathrm{strong}}(\mathtt{U},\mathtt{Y}) \subset \mathrm{Nev}_{\mathrm{strong}}(\mathtt{U},\mathtt{Y})$. Obviously, for multiplications we have $\mathrm{Nev}_+(\mathtt{U},\mathtt{Y})\,\mathrm{Nev}_+(\mathtt{X},\mathtt{U}) \subset \mathrm{Nev}_+(\mathtt{X},\mathtt{Y})$ and

$$\mathrm{Nev}_+(\mathtt{U},\mathtt{Y})\,\mathrm{Nev}_{+,\mathrm{strong}}(\mathtt{X},\mathtt{U}) \subset \mathrm{Nev}_{+,\mathrm{strong}}(\mathtt{X},\mathtt{Y}); \tag{6}$$

these two inclusions are valid with "+" removed too. The functions in $\mathrm{Nev}(\mathtt{U},\mathtt{Y})$ have $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued nontangential boundary functions in the strong sense [Mik08a] (unlike some in $\mathcal{H}^2_{\mathrm{strong}}(\mathtt{U})$ when $\dim \mathtt{U} = \infty$ [Mik06a, Section 4]).

In the previous sections we have several results on $\mathcal{H}^\infty$. Some of them can be extended to $\mathcal{H}^p$ by using the following lemma to reduce the extensions to the original results. The lemma says that a $\mathtt{U}$-valued $\mathcal{H}^p$ function f equals ϕg, where $\phi \in \mathcal{H}^p$ is scalar-valued and outer and $g \in \mathcal{H}^\infty$ is $\mathtt{U}$-valued and "inner".

Lemma 8.2 **($\mathcal{H}^p = \mathcal{H}^p(\mathbb{C})\,\mathcal{H}^\infty$)** *If $f \in \mathcal{H}^p(\mathtt{U})$, then $f = \phi g$, where $\phi \in \mathcal{H}^p(\mathbb{C})$ is outer and hence has no zeros and $g \in \mathcal{H}^\infty(\mathbb{C},\mathtt{U})$ has $\|g\| = 1$ a.e. on $\partial\mathbb{D}$.*

(This is [Nik02, Corollary 3.11.9], where ϕ is taken to be the Schwarz–Herglotz integral of the boundary function of $\|f\|_{\mathtt{U}}$ [Nik02, Theorem 3.9.1]. This ϕ is outer in the standard $\mathcal{H}^p$ sense; if $p \geq 2$, then an equivalent condition is that the closed span of the functions $z \mapsto \phi(z)z^n$ ($n \geq 0$) is dense in $\mathcal{H}^2(\mathbb{C})$.)

The set $\mathcal{H}^p_{\mathrm{strong}}(\mathtt{U},\mathtt{Y})$ coincides with the set of proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued functions N such that $Nu \in \mathcal{H}^p(\mathtt{Y})$ for every $u \in \mathtt{U}$, as one observes by modifying the proof of Lemma 8.1. It could be made a Banach space with the natural norm $\sup_{\|u\|_{\mathtt{U}}\leq 1} \|Nu\|_{\mathcal{H}^p}$, and $\mathcal{H}^p_{\mathrm{strong}}(\mathtt{U},\mathtt{Y}) = \mathcal{B}(\mathtt{U},\mathcal{H}^p(\mathtt{Y}))$ [Mik02, Lemma F.3.2(c&d)], but we do not need these facts. By the Uniform Boundedness Theorem, $\mathcal{H}^\infty_{\mathrm{strong}} = \mathcal{H}^\infty$. Obviously, we have $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y}), G \in \mathcal{H}^p_{\mathrm{strong}}(\mathtt{X},\mathtt{U}) \implies FG \in \mathcal{H}^p_{\mathrm{strong}}(\mathtt{X},\mathtt{Y})$. The spaces $\mathcal{H}^p_{\mathrm{strong}}(\mathbb{C};\mathtt{U})$ and $\mathcal{H}^p(\mathtt{U})$ can be identified.

By definition, weakly left-invertible functions "identify" $\mathcal{H}^2$ functions. They "identify" also Nev, $\mathrm{Nev}_{\mathrm{strong}}$, $\mathcal{H}^p$ and $\mathcal{H}^p_{\mathrm{strong}}$ functions:

Theorem 8.3 *Assume that $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is α-weakly left-invertible. If f is a α-proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function, then $Ff \in \mathrm{Nev}(\mathtt{X},\mathtt{Y}) \Leftrightarrow f \in \mathrm{Nev}(\mathtt{X},\mathtt{U})$.*

If V is a α-proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function, then $FV \in \mathrm{Nev}_{\mathrm{strong}}(\mathtt{X},\mathtt{Y}) \Leftrightarrow V \in \mathrm{Nev}_{\mathrm{strong}}(\mathtt{X},\mathtt{U})$.

The above also holds with Nev_+ in place of Nev as well as with $\mathcal{H}^p_{\mathrm{strong}}$ in place of $\mathrm{Nev}_{\mathrm{strong}}$. Finally, if f is an α-proper $\mathtt{U}$-valued function, then $Ff \in \mathcal{H}^p(\mathtt{Y}) \Leftrightarrow f \in \mathcal{H}^p(\mathtt{U})$.

Observe that $\mathtt{U} = \mathcal{B}(\mathbb{C},\mathtt{U})$.

Proof: 1° Let $Ff = \phi^{-1}g$ with $g \in \mathcal{H}^\infty(\mathtt{X},\mathtt{Y})$ and $\phi \in \mathcal{H}^\infty(\mathbb{C})$, $\phi \neq 0$ on $\mathbb{D}$. Then $F\phi f = g \in \mathcal{H}^\infty(\mathtt{X},\mathtt{Y})$, hence $\phi f \in \mathcal{H}^\infty(\mathtt{X},\mathtt{U})$, by Lemma 2.3. Consequently, $f = \phi^{-1}\phi f \in \mathrm{Nev}(\mathtt{X},\mathtt{U})$. The converse is trivial.

2° If $FVx \in \mathrm{Nev}(\mathbb{C},\mathtt{Y})\ \forall x \in \mathtt{X}$, then $Vx \in \mathrm{Nev}(\mathbb{C},\mathtt{U})\ \forall x \in \mathtt{X}$, by 1°, i.e., then $V \in \mathrm{Nev}_{\mathrm{strong}}(\mathtt{X},\mathtt{U})$. The converse is again trivial.

3° The proofs for Nev_+ are analogous. In the final claim, if $Ff \in \mathcal{H}^p(\mathtt{Y})$, then $Ff = \phi g$ as in Lemma 8.2, hence then $F\phi^{-1}f = g \in \mathcal{H}^\infty$, hence $\phi^{-1}f \in \mathcal{H}^\infty$, hence $f = \phi(\phi^{-1}f) \in \mathcal{H}^p$; the converse is again trivial. □

Now we shall extend Theorem 1.1 to these and other classes, i.e., we establish "weaker" equivalent forms of Theorem 1.2(i).

Theorem 8.4 *If $P = NM^{-1}$, where $N, M \in \mathrm{Nev}$ and $M(\alpha) \in \mathcal{GB}(\mathtt{U})$, then P has a α-w.r.c.f. $P = N_0M_0^{-1}$ and all such "Nev/Nev α-factorizations" are parameterized by*

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} V,\ V \in \mathrm{Nev}(\mathtt{U}),\ V(\alpha) \in \mathcal{GB}(\mathtt{U}). \tag{7}$$

If M is invertible on some open and connected $\Omega \subset \mathbb{D}$ and $\alpha \in \Omega$, then $M_0^{-1} = VM^{-1}$ on Ω (if $\dim \mathtt{U} < \infty$, then Ω need not be connected).

If $2 \le p \le \infty$, then the above also holds with $\mathcal{H}^p_{\mathrm{strong}}$ in place of Nev. The above also holds with Nev_+ in place of Nev.

Thus, a proper function P has a w.r.c.f. $\Leftrightarrow$ it has a "$\mathcal{H}^\infty / \mathcal{H}^\infty$ factorization" $\Leftrightarrow$ it has a "$\mathcal{H}^2_{\mathrm{strong}}/\mathcal{H}^2_{\mathrm{strong}}$ factorization" $\Leftrightarrow$ it has a "Nevanlinna/Nevanlinna factorization" (cf. Theorem 1.2).

Proof: 1° If $N = \phi^{-1}F$ and $M = \psi^{-1}G$ with $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$, $G \in \mathcal{H}^\infty(\mathtt{U})$, $M(\alpha) \in \mathcal{GB}(\mathtt{U})$, $\phi, \psi \in \mathcal{H}^\infty(\mathbb{C})$ and $\phi \neq 0$, $\psi \neq 0$ on $\mathbb{D}$, then $NM^{-1} = (\psi F)(\phi G)^{-1}$ is a right factorization, hence a w.r.c.f. $N_0M_0^{-1}$ exists, by Theorem 1.1.

2° Set $V := M_0^{-1}M$. Now $NM^{-1} = N_0M_0^{-1} = (N_0V)(M_0V)^{-1} = N_0VM^{-1}$ near α, hence $N = N_0V$. But $M = M_0V$ and $M, N \in \mathrm{Nev}$, hence $V \in \mathrm{Nev}(\mathtt{U})$, by Theorem 8.3 (since $\left[\begin{smallmatrix} N_0 \\ M_0 \end{smallmatrix}\right]$ is weakly left-invertible).

Conversely, the factorizations parameterized by (7) are obviously of the form $N, M \in \mathrm{Nev}$, $M(\alpha) \in \mathcal{GB}(\mathtt{U})$, $P = NM^{-1}$.

3° The claim below (7) follows, because $E := VM^{-1}$ is holomorphic on Ω and $M_0E = I = EM_0$ near α (if $\dim \mathtt{U} < \infty$, then the equation $M = M_0V$ implies that M_0 and V must be invertible on Ω even if it is not connected).

4° All of the above applies to Nev_+ in place of Nev too, mutatis mutandis, so only $\mathcal{H}^p_{\mathrm{strong}}$ remains to be treated.

5° The map $f \mapsto f \circ \phi_\alpha$, where $\phi_\alpha(z) := \frac{z+\alpha}{1+z\bar{\alpha}}$, maps α-proper functions to 0-proper functions, α-w.r.c.f.'s to 0-w.r.c.f.'s and $\mathcal{H}^p \to \mathcal{H}^p$, by [Mik07g, Lemma 6.7], hence also $\mathcal{H}^p_{\text{strong}} \to \mathcal{H}^p_{\text{strong}}$. Moreover, $f \circ \phi_\alpha \circ \phi_{-\alpha} = f$. Therefore, we may assume that $\alpha = 0$.

6° We thus assume that $P = NM^{-1}$ with $N, M \in \mathcal{H}^p_{\text{strong}}$, $M(0) \in \mathcal{GB}(\mathtt{U})$, $p \in [2, \infty]$.

6.1° Assume first that $P(0) = 0$ (hence $N(0) = 0$). By Lemma 2.22 we may assume that $P(0) = 0$ (hence $N(0) = 0$). We may assume that $M(0) = I$ (right-multiply N and M by $M(0)^{-1}$). Now we refer to the terminology of [Mik07g]. By [Mik07g, Theorem 5.5], an output-stable realization of $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is given by

$$\left(\begin{array}{c|c} A_\circlearrowleft & B \\ \hline C & 0 \\ F & 0 \end{array}\right) := \left(\begin{array}{c|c} S_L & z^{-1}\cdot \\ \hline \pi_0 N & 0 \\ \pi_0 M & I \end{array}\right). \tag{8}$$

Therefore, the system $\Sigma := \left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$, where $A := A_\circlearrowleft - BF$, is output-stabilizable (by F, with closed-loop system (8)), hence it satisfies the Finite Cost Condition. By [Mik07g, Lemma 3.3], it follows that the transfer function $\hat{\mathscr{D}}$ of Σ has a w.r.c.f. $\hat{\mathscr{D}} = N_0 M_0^{-1}$. As noted below [Mik07g, (7)], we have $\hat{\mathscr{D}} = NM^{-1} = P$.

6.2° Now if, instead, $P_0 := P(0) \neq 0$, we get a w.r.c.f. $(P - P_0) = NM^{-1}$ as above and then we can use Lemma 2.22 to obtain a w.r.c.f. of P

7° The remaining claims follow as in parts 2° and 3° above. □

By Theorem 8.3, α-weakly left-invertible functions (and w.r.c. pairs) "identify" the classes $\mathcal{H}^p$, $\mathcal{H}^p_{\text{strong}}$, Nev, Nev_+, $\text{Nev}_{\text{strong}}$ and $\text{Nev}_{+,\text{strong}}$. In the case of $\mathcal{H}^p$ or $\mathcal{H}^p_{\text{strong}}$ also the converse holds, i.e., that property can actually be used as the definition of α-weak left-invertibility:

Theorem 8.5 *Assume that $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ and $F(\alpha)$ is coercive. Then F is α-weakly left-invertible iff $Ff \in \mathcal{H}^p \implies f \in \mathcal{H}^p$ for every α-proper $\mathtt{U}$-valued function f.*

Assume also that $\mathtt{X} \neq \{0\}$. Then F is α-weakly left-invertible iff $FV \in \mathcal{H}^p_{\text{strong}} \implies V \in \mathcal{H}^p_{\text{strong}}$ for every α-proper $\mathcal{B}(\mathtt{X}, \mathtt{U})$-valued function V.

However, the above is not true for Nev, Nev_+, $\text{Nev}_{\text{strong}}$ or $\text{Nev}_{+,\text{strong}}$. Indeed, $F(z) = z - 1/2$ is not 0-weakly left-invertible but it satisfies $Ff \in \text{Nev} \implies f \in \text{Nev}$ (because here $f = (z - 1/2)^{-1} Ff \in \text{Nev}$).

Proof: 1° The "only if" claims are from Theorem 8.3, so assume first that $Ff \in \mathcal{H}^p(\mathtt{Y}) \implies f \in \mathcal{H}^p(\mathtt{U})$ for every α-proper $\mathtt{U}$-valued f.

With $F = F_w F_r$ from Theorem 2.10(a) we have $FF_r^{-1} f = F_w f \in \mathcal{H}^p(\mathtt{Y})$, hence $F_r^{-1} f \in \mathcal{H}^p(\mathtt{U})$, for every $f \in \mathcal{H}^p(\mathtt{U})$. By Lemma 8.1, $F_r^{-1} \in \mathcal{H}^\infty$, hence also $F = F_w F_r$ is α-weakly left-invertible.

2° Assume then that $FV \in \mathcal{H}^p_{\text{strong}}(\mathtt{X}, \mathtt{Y}) \implies V \in \mathcal{H}^p_{\text{strong}}(\mathtt{X}, \mathtt{U})$ for every α-proper $\mathcal{B}(\mathtt{X}, \mathtt{U})$-valued V. Pick $x_0 \in \mathtt{X}$, $\Lambda \in \mathtt{X}^*$ such that $\Lambda x_0 = 1$. Given any α-proper $\mathtt{U}$-valued f such that $Ff \in \mathcal{H}^p(\mathtt{Y})$, set $V := f\Lambda$ to have $FVx =$

$Ff\Lambda x \in \mathcal{H}^p\ \forall x \in \mathtt{X}$, hence $V \in \mathcal{H}^p_{\text{strong}}(\mathtt{X},\mathtt{U})$, hence $f = Vx_0 \in \mathcal{H}^p(\mathtt{U})$. Because f was arbitrary, F is α-weakly left-invertible, by 1°. □

If $F \in \text{Nev}(\mathtt{U},\mathtt{Y})$ and $F(\alpha)$ is coercive, then $F = F_w F_r$, where $F_w \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is inner and α-weakly left-invertible, $F_r \in \text{Nev}(\mathtt{U})$ and $F_r(\alpha) \in \mathcal{GB}(\mathtt{U})$. This and more is listed below:

Theorem 8.6 ($\mathbf{F=F}_w\mathbf{F}_r$) *Theorem 2.10(a)&(c) and Lemma 2.11 hold also with "$F \in \text{Nev}$" in place of "$F \in \mathcal{H}^\infty$" and "$F_r \in \text{Nev}$" in place of "$F_r \in \mathcal{H}^\infty$".*

The above also holds with Nev_+ *or* $\mathcal{H}^p_{\text{strong}}$ *in place of* Nev *if* $p \in [2,\infty]$.

Proof: Theorem 2.10(a)&(c) for Nev or Nev_+ follows by applying Theorem 2.10 to G, where $F = \phi^{-1}G$ as in the definition of Nev or Nev_+, and then replacing F_r by $\phi^{-1}F_r$. For $\mathcal{H}^p_{\text{strong}}$ one can use the original proof from [Mik07g], mutatis mutandis, with Theorem 8.4 instead of Theorem 2.1.

Now the two references to Theorem 2.10(a) in the proof of Lemma 2.11 can be replaced by references to the modified Theorem 2.10(a) (described in Theorem 8.6). □

Notes

We have not made any separability assumptions. Actually, [RR85] and [Nik02] do, so one should consult [Mik09] or [Mik07e] instead in the general case.

Note that not all $\mathcal{H}^2_{\text{strong}}$ functions are contained in the Nevanlinna class, indeed, they need not even have $\mathcal{B}$-valued boundary functions [Mik06a] (which can be transformed to a discrete-time counter-example by a weighted Cayley transform), unlike those in the Nevanlinna class. Here we assumed that U and Y are infinite-dimensional; the opposite inclusion does not hold even in the scalar-valued case.

Theorems 8.3, 8.4 and 8.6 would hold even if, in the definition of Nev, we dropped the requirement that $\phi \neq 0$ on $\mathbb{D}$; the only exception would be that we should assume in Theorem 8.4 that P is proper (i.e., that N is proper; note also that then $M_0^{-1} = VM^{-1}$ might not exist at some isolated points of Ω). The same applies to almost any another reasonable alternative assumption on ϕ except that Theorem 8.4 is lost in some cases (e.g., if we only require that ϕ is meromorphic on a neighborhood of the origin; even in that case Theorems 8.3 and 8.6 would hold).

Theorem 8.5 gives an alternative proof of the fact that the classical "$\mathcal{H}^\infty$ w.r.c." is the same as our "$\mathcal{H}^2$ w.r.c.", i.e., that the classical weak coprimeness implies the (equivalent) variant used in this article. However, it does not prove the existence of either factorization (in the operator-valued case).

Not everything in Section 2 can be extended to, e.g., Nev in place of $\mathcal{H}^\infty$: unlike in Theorem 2.8(a), we may have, in terms of Theorem 8.4, (scalar-valued) $V = N \in \mathcal{H}^2 \setminus \mathcal{H}^\infty$, $V^{-1} \in \mathcal{H}^\infty$, $M = I$, $N_0 = I$, $M_0 = V^{-1}$, so that M_0^{-1} need not be uniformly bounded on Ω even if M^{-1} were (e.g., take $\Omega = \mathbb{D}$).

Further discrete-time results are given near Remark 7.6 (and in [Mik07g]).

9 $\mathcal{H}^2_{\rm strong}$ and Nevanlinna w.r.c.f.'s

Here we establish the results in the previous section in the continuous-time setting.

Throughout this section we assume that $1 \leq p \leq \infty$ and $\alpha \in \mathbb{C}^+$. By $\mathcal{H}^p(\mathtt{U})$ we denote (the Banach space of) those holomorphic $f : \mathbb{C}^+ \to \mathtt{U}$ for which $\|f\|_{\mathcal{H}^p} := \sup_{r>0} \|f(r+i\cdot)\|_{\mathrm{L}^p} < \infty$. We define the classes $\mathcal{H}^p_{\rm strong}$, Nev, Nev_+, $\mathrm{Nev}_{\rm strong}$ and $\mathrm{Nev}_{+,\rm strong}$ as in Section 8 but with the half-plane $\mathbb{C}^+$ in place of the unit disc $\mathbb{D}$.

These classes on $\mathbb{C}^+$ correspond one-to-one and onto to the corresponding classes on $\mathbb{D}$:

Lemma 9.1 (Cayley) *Let P be an α-proper function. We have $P \in \mathcal{H}^p$ iff $z \mapsto w_p(z)P(c(z)) =: P_c$ is $\mathcal{H}^p$ over the unit disc, where $c(z) := (\alpha - \bar{\alpha}z)/(1+z)$ and $w_p(z) := (1+z)^{-2/p}$. The same holds with $\mathcal{H}^p_{\rm strong}$ in place of $\mathcal{H}^p$. The conformal map c maps $\mathbb{D} \to \mathbb{C}^+$ and $\partial\mathbb{D} \to i\mathbb{R} \cup \{\infty\}$ one-to-one and onto and $c(0) = \alpha$.*

Moreover, $P \in$ Nev iff $P \circ c$ is Nev over the unit disc; the same holds with Nev_+, $\mathrm{Nev}_{\rm strong}$, $\mathrm{Nev}_{+,\rm strong}$, $\mathcal{H}^\infty$ or "scalar outer" in place of Nev. Similarly, $P \mapsto P \circ c$ maps α-proper (resp., α-weakly left-invertible, α-w.r.c.) functions to "unit disc" 0-proper (resp., 0-weakly left-invertible, 0-w.r.c.) functions, bijectively.

(The proof is analogous to the well-known scalar case (in the $\mathcal{H}^p$ case; the other claims follow) and hence omitted.)

So, e.g., NM^{-1} is an α-w.r.c.f. iff $(N \circ c)(M \circ c)^{-1}$ is a "unit disc" 0-w.r.c.f.

Using Lemma 9.1, we obtain the following:

Theorem 9.2 *All results of Section 8 hold also in this continuous-time terminology with "$\mathbb{C}^+$" in place of "$\mathbb{D}$".*

For reference purposes, we repeat those results in Lemma 9.4—Theorem 9.9 below.

We note that, in particular, a function $P \in \mathcal{H}^\infty_\omega(\mathtt{U}, \mathtt{Y})$ (for some $\omega \geq 0$) has a w.r.c.f. iff it has such a "Nevanlinna/Nevanlinna factorization" (as described in Theorem 8.4) or a "$\mathcal{H}^p_{\rm strong}/\mathcal{H}^p_{\rm strong}$ factorization" for some (hence any) $\alpha \in \mathbb{C}^+_\omega$ (and $p \geq 2$), by Theorem 3.1(b). Thus, we have two more equivalent conditions to Theorem 1.2(i). Recall also those from Theorems 6.2 and 6.3.

Also the comments between the results in Section 8 hold except that a holomorphic function $f : \mathbb{C}^+ \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ is in $\mathrm{Nev}(\mathtt{U}, \mathtt{Y})$ iff $\sup_{x>0} \int_{\mathbb{R}} ((x+1)^2 + y^2)^{-1} \log^+ \|f(x+iy)\|\, dy < \infty$ (instead of $\sup_{0<r<1} \log^+ \|f(re^{i\cdot})\|_1 < \infty$).

Theorems 3.1(b) and 3.3 show how the α-w.r.c.f. or α-weak left-invertibility provided by the above results is often a (proper) w.r.c.f. or weak left-invertibility. Also Lemma 8.1 and Theorem 8.3 apply to "proper" concepts too but not, e.g., Theorem 8.4, as illustrated below.

Example 9.3 The function $P(s) := N(s) := s + 1 \in \mathrm{Nev}_+(\mathbb{C})$ satisfies $P = NI^{-1}$ but P is not proper and hence it does not have a w.r.c.f. However,

P has the α-r.c.f. $P(s) = IM^{-1}$, where $M(s) := (s+1)^{-1} \in \mathrm{Nev}_+(\mathbb{C}) \cap \mathcal{H}^p(\mathbb{C})$ if $p > 1$. $\triangleleft$

Lemma 9.4 *If V is an α-proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function, $1 \le p \le \infty$ and $Vf \in \mathcal{H}^p(\mathtt{U})$ for each $f \in \mathcal{H}^p(\mathtt{X})$, then $V \in \mathcal{H}^\infty(\mathtt{X},\mathtt{U})$.*

Lemma 9.5 ($\mathcal{H}^p = \mathcal{H}^p(\mathbb{C})\,\mathcal{H}^\infty$) *If $f \in \mathcal{H}^p(\mathtt{U})$, then $f = \phi g$, where $\phi \in \mathcal{H}^p(\mathbb{C})$ is outer and hence has no zeros and $g \in \mathcal{H}^\infty(\mathbb{C},\mathtt{U})$ has $\|g\| = 1$ a.e. on $\partial\mathbb{D}$.*

Theorem 9.6 *Assume that $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is α-weakly left-invertible. If f is a α-proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function, then $Ff \in \mathrm{Nev}(\mathtt{X},\mathtt{Y}) \Leftrightarrow f \in \mathrm{Nev}(\mathtt{X},\mathtt{U})$. If V is a α-proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function, then $FV \in \mathrm{Nev}_{\mathrm{strong}}(\mathtt{X},\mathtt{Y}) \Leftrightarrow V \in \mathrm{Nev}_{\mathrm{strong}}(\mathtt{X},\mathtt{U})$.*

The above also holds with Nev_+ in place of Nev as well as with $\mathcal{H}^p_{\mathrm{strong}}$ in place of $\mathrm{Nev}_{\mathrm{strong}}$. Finally, if f is an α-proper $\mathtt{U}$-valued function, then $Ff \in \mathcal{H}^p(\mathtt{Y}) \Leftrightarrow f \in \mathcal{H}^p(\mathtt{U})$.

Theorem 9.7 *If $P = NM^{-1}$, where $N, M \in \mathrm{Nev}$ and $M(\alpha) \in \mathcal{GB}(\mathtt{U})$, then P has a α-w.r.c.f. $P = N_0 M_0^{-1}$ and all such "Nev/Nev α-factorizations" are parameterized by*

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} V,\ V \in \mathrm{Nev}(\mathtt{U}),\ V(\alpha) \in \mathcal{GB}(\mathtt{U}). \tag{9}$$

If M is invertible on some open and connected $\Omega \subset \mathbb{C}^+$ and $\alpha \in \Omega$, then $M_0^{-1} = VM^{-1}$ on Ω (if $\dim \mathtt{U} < \infty$, then Ω need not be connected).

If $2 \le p \le \infty$, then the above also holds with $\mathcal{H}^p_{\mathrm{strong}}$ in place of Nev. The above also holds with Nev_+ in place of Nev.

Theorem 9.8 *Assume that $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ and $F(\alpha)$ is coercive. Then F is α-weakly left-invertible iff $Ff \in \mathcal{H}^p \implies f \in \mathcal{H}^p$ for every α-proper $\mathtt{U}$-valued function f.*

Assume also that $\mathtt{X} \ne \{0\}$. Then F is α-weakly left-invertible iff $FV \in \mathcal{H}^p_{\mathrm{strong}} \implies V \in \mathcal{H}^p_{\mathrm{strong}}$ for every α-proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function V.

Theorem 9.9 ($\mathbf{F}=\mathbf{F}_w\mathbf{F}_r$) *Theorem 2.10(a)&(c) and Lemma 2.11 hold also with "$F \in \mathrm{Nev}$" in place of "$F \in \mathcal{H}^\infty$" and "$F_r \in \mathrm{Nev}$" in place of "$F_r \in \mathcal{H}^\infty$".*

The above also holds with Nev_+ or $\mathcal{H}^p_{\mathrm{strong}}$ in place of Nev if $p \in [2,\infty]$.

Also Theorem 6.2 contains an existence result similar to Theorem 9.7.

10 Conclusions

We have established the existence of a w.r.c.f. in the operator-valued case (Theorem 1.1, assuming a right factorization; formally even weaker assumptions were treated in Sections 6–9). As such, the result is new even in the matrix-valued case, in which the equivalent "gcd = 1" factorization was established already in [Ino88] and [Smi89]. In Sections 2 and 3 we showed that all competing definitions of w.r.c.f.'s of proper functions are equivalent. We also explained the relations of these definitions in the general case, in which substantial differences were found (Section 4).

In Theorem 1.2 we showed how the w.r.c.f.'s relate to the stabilizability of control systems. Further details are given in Section 5 and in [Mik06b] (or in [Mik07g], in discrete time).

On this level the theory becomes almost as neat as for rational functions, although the computation of the LQ-optimal control and the Riccati equations become highly technical [Mik06b] [OC04]. Moreover, the conditions in Theorems 1.2 are not equivalent to those in 1.3, hence the chain of interconnected properties is divided in two in the case of non-rational transfer functions (or of infinite-dimensional systems).

Furthermore, several new useful properties of weakly coprime (or weakly left-invertible) functions and their Hankel operators were found, as well as necessary and sufficient conditions for the existence of a w.r.c.f. Also some related results on r.c.f.'s were given.

A Cayley transform

In this appendix we list the system-theoretic properties of the Cayley transform that were needed in some of the above proofs. The signal-theoretic properties were given in Lemma 9.1.

Recall the definition of a WPLS etc. from Section 5.

The Cayley transform maps WPLSs to a subset of discrete-time systems and "preserves" most of their properties. Part of this is illustrated below.

Lemma A.1 *Let* $\Sigma = \left[\begin{smallmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$ *be a WPLS on* $(\mathtt{U},\mathtt{X},\mathtt{Y})$*, and let* $\alpha\in\mathbb{C}$ *be such that* $\operatorname{Re}\alpha > \max\{0,\omega_A\}$*, where* $\omega_A := \inf_{t>0}[t^{-1}\log\|\mathscr{A}^t\|]$.

(a) *Then* $A_{\mathrm{d}} := (\bar\alpha + A)(\alpha - A)^{-1} = -I + 2(\operatorname{Re}\alpha)(\alpha-A)^{-1}\in\mathcal{B}(\mathtt{X})$, $B_{\mathrm{d}} := \sqrt{2\operatorname{Re}\alpha}\,(\alpha-A)^{-1}B\in\mathcal{B}(\mathtt{U},\mathtt{X})$, $C_{\mathrm{d}} := \sqrt{2\operatorname{Re}\alpha}\,C(\alpha-A)^{-1}\in\mathcal{B}(\mathtt{X},\mathtt{Y})$, *and* $D_{\mathrm{d}} := \hat{\mathscr{D}}(\alpha)\in\mathcal{B}(\mathtt{U},\mathtt{Y})$ *determine a discrete-time system* $\Sigma_{\mathrm{d}} := \left(\begin{smallmatrix}A_{\mathrm{d}} & B_{\mathrm{d}}\\ C_{\mathrm{d}} & D_{\mathrm{d}}\end{smallmatrix}\right)$ *called* the Cayley transform *of* Σ. *Conversely,* Σ_{d} *determines* Σ *uniquely. We also have* $\operatorname{Ker}(A_{\mathrm{d}}+I)=\{0\}$*, and* $\operatorname{Dom}(A) = \operatorname{Ran}(A_{\mathrm{d}}+I)$.

(b) *If we discretize the dual* Σ^{d} *of* Σ*, we obtain the* dual *system* $\left(\begin{smallmatrix}A_{\mathrm{d}}^* & C_{\mathrm{d}}^*\\ B_{\mathrm{d}}^* & D_{\mathrm{d}}^*\end{smallmatrix}\right)$ *of* Σ_{d}. *(I.e.,* $(\Sigma^{\mathrm{d}})_{\mathrm{d}} = (\Sigma_{\mathrm{d}})^{\mathrm{d}}$*.)*

(c) *For each* $s\in\rho(A)\setminus\{-\bar\alpha\}$ *(or equivalently,* $z^{-1}\in\rho(A_{\mathrm{d}})\setminus\{-1\}$*, where* $z := (\alpha - s)/(\bar\alpha+s)$ *or* $s = (\alpha - \bar\alpha z)/(1+z)$*), we have* $\hat{\mathscr{D}}(s) = \hat{\mathscr{D}}_{\mathrm{d}}(z)$*, where* $\hat{\mathscr{D}}_{\mathrm{d}}(z) := D_{\mathrm{d}} + zC_{\mathrm{d}}(I - zA_{\mathrm{d}})^{-1}B_{\mathrm{d}}$.

(d) Σ *satisfies the continuous-time Finite Cost Condition iff* Σ_{d} *satisfies the discrete-time Finite Cost Condition.*

Note that the map $s \mapsto z$ in (c) is one-to-one and onto $\mathbb{C}^+ \to \mathbb{D}$.

Proof: Use [Sta05], particularly Proposition 12.2.1 (with $\beta := \bar{\alpha},\ \gamma := -1$) and Theorem 12.3.5 to obtain (a) (note that Staffans' z is our z^{-1}). Part (b) is straightforward. From [OC04, Lemma 8 and Theorems 9&10] we get (c)&(d). □

We mention here a very important property that is not needed in this article. The WPLS Σ is called *externally stable* (or "system stable") if $\hat{\mathscr{D}} \in \mathcal{H}^\infty$ and $\mathscr{C}x_0, \mathscr{B}^*x_0 \in \mathrm{L}^2$ for every $x_0 \in \mathtt{X}$ (analogously for discrete-time systems, with ℓ^2 in place of L^2). One could also show that any externally stable extension of Σ_d is the Cayley transform of a (unique) externally stable WPLS and that admissible state-feedback pairs for Σ are mapped to admissible state-feedback pairs for Σ_d (also the corresponding closed-loop systems are mapped to each other).

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