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Abstract: *A new a posteriori error estimate is derived for the stationary convection-reaction-diffusion equation. In order to estimate the approximation error in the usual energy norm, the underlying bilinear form is decomposed into a computable integral and two other terms which can be estimated from above using elementary tools of functional analysis. Two auxiliary parameter-functions are introduced to construct such a splitting and tune the resulting bound. If these functions are chosen in an optimal way, the exact energy norm of the error is recovered, which proves that the estimate is sharp. The presented methodology is completely independent of the numerical technique used to compute the approximate solution. In particular, it is applicable to approximations which fail to satisfy the Galerkin orthogonality, e.g. due to an inconsistent stabilization, flux limiting, low-order quadrature rules, round-off and iteration errors etc. Moreover, the only constant that appears in the proposed error estimate is global and stems from the Friedrichs-Poincaré inequality. Numerical experiments illustrate the potential of the proposed error estimation technique.*

AMS subject classifications: 65N15, 65N50, 76M30

Keywords: convection-reaction-diffusion, a posteriori error estimation, adaptivity

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1 Introduction

Many mathematical models are based on (systems of) convection-reaction-diffusion equations which need to be discretized and solved numerically. The goal of *a posteriori* error estimation is to quantify the discrepancy between the exact and the numerical solution of the problem at hand. Currently, reliable error control is feasible, e.g., for finite element approximations of the Poisson equation and similar elliptic problems. At the same time, there is still a lot of room for research in the field of error estimation for convection-diffusion equations and hyperbolic conservation laws, although significant advances were achieved during the past two decades, see, e.g., [2, 4, 6, 7, 12].

An inherent limitation of many a posteriori error estimation techniques is the presence of dubious constants which are difficult to estimate (cf. [1]). The uncertainty involved in the computation of these constants may seriously reduce the practical utility of the resulting estimates. Moreover, some popular methods rely on the existence of an equivalent minimization problem or assume the Galerkin orthogonality. For the residual to be orthogonal to the space of test functions, the discretization must be performed by a consistent (Petrov-)Galerkin method and the resulting algebraic equations must be solved exactly. These requirements can rarely be satisfied in practice because of numerical quadrature, round-off errors, slack tolerances for iterative solvers and even programming bugs. The use of upwinding or flux/slope limiters in finite element codes may also violate the Galerkin orthogonality.

A promising general approach to error estimation for elliptic problems was introduced by Repin *et al.* [9, 10, 11]. In the present paper, a simplified version of this methodology [5, 8] is extended to stationary convection-diffusion equations. The resulting upper bound for the error in the energy norm is shown be sharp if the involved parameter-functions are chosen in an optimal way. Moreover, there is just one global constant which depends solely on the geometry of the domain and does not change in the course of mesh adaptation. The derivation of the new estimate and the proof of optimality are followed by a discussion of practical implementation details. Finally, numerical experiments are performed for a 1D test problem with a known analytical solution.

2 Problem statement

Consider the stationary convection-reaction-diffusion problem

$$\begin{cases} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^d,\ d \geq 1$, is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. As usual, it is assumed that $\varepsilon > 0,\ \mathbf{b} \in \mathbf{W}^1_\infty(\Omega),\ c \in L_\infty(\Omega)$.

The weak form of the above problem reads: Find $u \in H^1_0(\Omega)$ such that

$$a(u, w) = F(w), \qquad \forall w \in H^1_0(\Omega), \tag{2}$$

where the bilinear form $a(\cdot,\cdot)$ and the linear functional $F(\cdot)$ are given by

$$a(u,w) = \int\limits_\Omega \varepsilon\nabla u\cdot\nabla w\,dx + \int\limits_\Omega \mathbf{b}\cdot\nabla u\, w\; dx + \int\limits_\Omega cuw\; dx \tag{3}$$

$$F(w) = \int\limits_\Omega fw\; dx, \qquad u,w\in H_0^1(\Omega). \tag{4}$$

The so-defined $a(\cdot,\cdot)$ is coercive provided that $c-\frac{1}{2}\nabla\cdot\mathbf{b}\geq 0$. Indeed,

$$\begin{aligned} a(w,w) &= \int\limits_\Omega \varepsilon\nabla w\cdot\nabla w\,dx + \int\limits_\Omega (\mathbf{b}\cdot\nabla w)\, w\,dx + \int\limits_\Omega cw^2\,dx \\ &= \varepsilon\int\limits_\Omega |\nabla w|^2\,dx + \int\limits_\Omega \left(c-\frac{1}{2}\nabla\cdot\mathbf{b}\right) w^2\,dx \geq C_\epsilon\|w\|^2_{1,\Omega}, \end{aligned} \tag{5}$$

where C_ϵ is a positive constant and $\|\cdot\|_{1,\Omega}$ is the standard norm in $H^1(\Omega)$. Thus, the unique solvability of (2) follows from the Lax-Milgram lemma.

3 Error estimation

Let $\bar{u}$ be a function from $H_0^1(\Omega)$ which is supposed to be an approximate solution of problem (2) but there are no restrictions on the numerical method to be used. The error $e := u-\bar{u}$ will be estimated in the energy norm

$$|\!|\!| e |\!|\!|^2_\Omega := \varepsilon\int\limits_\Omega |\nabla e|^2\; dx + \int\limits_\Omega \left(c-\frac{1}{2}\nabla\cdot\mathbf{b}\right)\, e^2\; dx = a(e,e). \tag{6}$$

Using (2) with test function $w=u-\bar{u}$, we obtain the following representation

$$\begin{aligned} a(e,e) &= \varepsilon\int\limits_\Omega \nabla(u-\bar{u})\cdot\nabla(u-\bar{u})\; dx + \int\limits_\Omega \mathbf{b}\cdot\nabla(u-\bar{u})(u-\bar{u})\; dx \\ &+ \int\limits_\Omega c(u-\bar{u})\,(u-\bar{u})\; dx = \int\limits_\Omega f(u-\bar{u})\; dx - \varepsilon\int\limits_\Omega \nabla\bar{u}\cdot\nabla(u-\bar{u})\; dx \\ &- \int\limits_\Omega \mathbf{b}\cdot\nabla\bar{u}\,(u-\bar{u})\; dx - \int\limits_\Omega c\,\bar{u}\,(u-\bar{u})\; dx. \end{aligned} \tag{7}$$

Furthermore, it is worthwhile to regroup some terms in relation (7) and introduce an auxiliary vector function $\mathbf{y}^\star\in H(div,\Omega)$ so that

$$\begin{aligned} a(u-\bar{u},u-\bar{u}) &= \int\limits_\Omega [f-\mathbf{b}\cdot\nabla\bar{u}-c\,\bar{u}](u-\bar{u})\; dx - \int\limits_\Omega \mathbf{y}^\star\cdot\nabla(u-\bar{u})\; dx \\ &+ \int\limits_\Omega [\mathbf{y}^\star-\varepsilon\nabla\bar{u}]\cdot\nabla(u-\bar{u})\; dx = \int\limits_\Omega [f-\mathbf{b}\cdot\nabla\bar{u}-c\,\bar{u}](u-\bar{u})\; dx \\ &+ \int\limits_\Omega \nabla\cdot\mathbf{y}^\star(u-\bar{u})\; dx + \int\limits_\Omega [\mathbf{y}^\star-\varepsilon\nabla\bar{u}]\cdot\nabla(u-\bar{u})\; dx \end{aligned} \tag{8}$$

after integration by parts for the second term. Finally, let us introduce another auxiliary function $v \in H_0^1(\Omega)$ and consider the following decomposition

$$a(u-\bar{u}, u-\bar{u}) = I_1 + I_2 + I_3, \tag{9}$$

where the terms $I_1,\ I_2$ and I_3 are defined as follows

$$\begin{aligned} I_1 &= \int\limits_\Omega \left[f - \mathbf{b}\cdot\nabla\bar{u} - c\bar{u} + \nabla\cdot(\mathbf{y}^\star - \mathbf{b}v) + cv\right](u-\bar{u})\ dx, && (10)\\ I_2 &= \int\limits_\Omega \left[\mathbf{y}^\star - \varepsilon\nabla(\bar{u}-v)\right]\cdot\nabla(u-\bar{u})\ dx, && (11)\\ I_3 &= \int\limits_\Omega \left[(\nabla\cdot(\mathbf{b}v) - c\,v)(u-\bar{u}) - \varepsilon\nabla v\cdot\nabla(u-\bar{u})\right]\ dx. && (12)\end{aligned}$$

Integration by parts using Green's formula reveals that

$$\begin{aligned} I_3 &= \int\limits_\Omega \left[v(\mathbf{b}\cdot\nabla\bar{u} + c\,\bar{u}) + \varepsilon\nabla v\cdot\nabla\bar{u}\right]\ dx - \int\limits_\Omega \left[v(\mathbf{b}\cdot\nabla u + c\,u) + \varepsilon\nabla v\cdot\nabla u\right]\ dx\\ &= \int\limits_\Omega \left[v(\mathbf{b}\cdot\nabla\bar{u} + c\,\bar{u} - f) + \varepsilon\nabla v\cdot\nabla\bar{u}\right]\ dx = a(\bar{u}, v) - F(v) = R(v,\bar{u}), \quad (13)\end{aligned}$$

where $R(v,\bar{u})$ is the residual of problem (2) for $w = v$ and $\bar{u}$ in place of u.

Hence, the term I_3 is computable and it remains to derive an upper bound for the integrals I_1 and I_2. The Cauchy-Schwarz inequality yields

$$I_1 \leq \|f - \mathbf{b}\cdot\nabla\bar{u} - c\bar{u} + \nabla\cdot(\mathbf{y}^\star - \mathbf{b}v) + cv\|_{0,\Omega}\|u-\bar{u}\|_{0,\Omega}. \tag{14}$$

Due to the Friedrichs-Poincaré inequality $\|w\|_{0,\Omega} \leq C_\Omega\|\nabla w\|_{0,\Omega},\ \forall w \in H_0^1(\Omega)$, where C_Ω is a positive constant and $\|\cdot\|_{0,\Omega}$ is the L_2-norm, we have

$$I_1 \leq C_\Omega\|f - \mathbf{b}\cdot\nabla\bar{u} - c\bar{u} + \nabla\cdot(\mathbf{y}^\star - \mathbf{b}v) + cv\|_{0,\Omega}\|\nabla(u-\bar{u})\|_{0,\Omega}. \tag{15}$$

Similarly, the Cauchy-Schwarz inequality yields the estimate

$$I_2 \leq \|\mathbf{y}^\star - \varepsilon\nabla(\bar{u}-v)\|_{0,\Omega}\|\nabla(u-\bar{u})\|_{0,\Omega}. \tag{16}$$

Combining inequalities (15) and (16) we obtain an estimate of the form

$$I_1 + I_2 \leq \Lambda(v, \mathbf{y}^\star, \bar{u})\|\nabla(u-\bar{u})\|_{0,\Omega}, \tag{17}$$

where the functional $\Lambda(v, \mathbf{y}^\star, \bar{u})$ is given by the relation

$$\begin{aligned} \Lambda(v, \mathbf{y}^\star, \bar{u}) &= C_\Omega\|f - \mathbf{b}\cdot\nabla\bar{u} - c\bar{u} + \nabla\cdot(\mathbf{y}^\star - \mathbf{b}v) + cv\|_{0,\Omega}\\ &+ \|\mathbf{y}^\star - \varepsilon\nabla(\bar{u}-v)\|_{0,\Omega}. \end{aligned} \tag{18}$$

The Young inequality implies that for any $p \geq 0$ and $q \geq 0$

$$pq \leq \frac{\sigma}{2}p^2 + \frac{1}{2\sigma}q^2, \qquad \sigma > 0. \tag{19}$$

Consider $pq = \Lambda(v, \mathbf{y}^\star, \bar{u}) \|\nabla(u-\bar{u})\|_{0,\Omega}$, where $p := \sqrt{\Lambda(v, \mathbf{y}^\star, \bar{u})} \|\nabla(u-\bar{u})\|_{0,\Omega}$ and $q := \sqrt{\Lambda(v, \mathbf{y}^\star, \bar{u})}$. This enables us to estimate the right-hand side of (17) in terms of the energy norm (6) which resides in the left-hand side of (9)

$$I_1 + I_2 \leq \frac{\sigma}{2}\Lambda(v, \mathbf{y}^\star, \bar{u}) \|\nabla(u - \bar{u})\|_{0,\Omega}^2 + \frac{1}{2\sigma}\Lambda(v, \mathbf{y}^\star, \bar{u}) \tag{20}$$

$$\leq \frac{\sigma}{2\varepsilon}\Lambda(v, \mathbf{y}^\star, \bar{u}) \|\!|u - \bar{u}|\!\|_\Omega^2 + \frac{1}{2\sigma}\Lambda(v, \mathbf{y}^\star, \bar{u}). \tag{21}$$

Finally, we substitute this inequality into (9) and recall (13)

$$\|\!|u - \bar{u}|\!\|_\Omega^2 \leq \frac{\sigma}{2\varepsilon}\Lambda(v, \mathbf{y}^\star, \bar{u}) \|\!|u - \bar{u}|\!\|_\Omega^2 + \frac{1}{2\sigma}\Lambda(v, \mathbf{y}^\star, \bar{u}) + R(v, \bar{u}). \tag{22}$$

Thus, the energy norm of the error is bounded from above by

$$\|\!|u - \bar{u}|\!\|_\Omega^2 \leq \frac{R(v, \bar{u}) + \frac{1}{2\sigma}\Lambda(v, \mathbf{y}^\star, \bar{u})}{1 - \frac{\sigma}{2\varepsilon}\Lambda(v, \mathbf{y}^\star, \bar{u})}, \tag{23}$$

where the free parameter $\sigma > 0$ is to be chosen so that

$$\frac{\sigma}{2\varepsilon}\Lambda(v, \mathbf{y}^\star, \bar{u}) < 1. \tag{24}$$

Using **EST** to denote the (computable) right-hand side of (23), the upper bound for the energy norm of the error can be written as follows

$$\|\!|e|\!\|_\Omega^2 \leq \mathbf{EST}(\sigma, \mathbf{y}^\star, v, \bar{u}). \tag{25}$$

Recall that estimate (25) is valid for an arbitrary choice of $\mathbf{y}^\star \in H(div, \Omega)$, $v \in H_0^1(\Omega)$ and $\sigma > 0$ satisfying (24). Clearly, these parameters should be designed so as to minimize the functional **EST** as far as possible. Let the corresponding optimal values be denoted by $\mathbf{y}^\star_{opt}$, v_{opt} and σ_{opt}, respectively. In the next section we will show that the optimal upper bound

$$\underline{\mathbf{EST}} := \mathbf{EST}(\sigma_{opt}, \mathbf{y}^\star_{opt}, v_{opt}, \bar{u}) \tag{26}$$

reduces to the energy norm (6), which means that estimate (25) is sharp.

Remark. If the diffusion coefficient ε is small as compared to $|\mathbf{b}|$, then the standard energy norm does not provide a proper control of the error. A possible remedy is to add some streamline diffusion to the weak formulation (2) even if the approximate solution $\bar{u}$ is computed using a different stabilization technique such as finite volume upwinding or some sort of flux correction.

4 Sharpness of the estimate

In order to prove that (25) holds as equality for certain values of $\mathbf{y}^\star$, v and σ, let us consider the weak solution $v \in H_0^1(\Omega)$ of the adjoint problem

$$a^*(v, w) = R(w, \bar{u}), \qquad \forall w \in H_0^1(\Omega), \tag{27}$$

where $a^*(\cdot,\cdot)$ is a bilinear form such that $a^*(v,w) = a(w,v)$, i.e.,

$$a^*(v,w) = \int_\Omega \varepsilon \nabla v \cdot \nabla w \, dx - \int_\Omega [\nabla \cdot (\mathbf{b} v) - c\, v] w \, dx. \tag{28}$$

The linear functional $R(w,\bar{u}) = a(\bar{u},w) - F(w)$ represents the residual of the primal problem (2) evaluated using $\bar{u}$ instead of u. That is,

$$R(w,\bar{u}) = \int_\Omega \varepsilon \nabla \bar{u} \cdot \nabla w \, dx + \int_\Omega [\mathbf{b} \cdot \nabla \bar{u} + c\,\bar{u} - f] w \, dx. \tag{29}$$

Furthermore, let us define the free parameter $\mathbf{y}^\star$ as follows

$$\mathbf{y}^\star = \varepsilon \nabla(\bar{u} - v). \tag{30}$$

Importantly, the so-defined $\mathbf{y}^\star$ does belong to the space $H(div,\Omega)$ because our weak adjoint problem (27) can be represented in the following form

$$\int_\Omega \varepsilon \nabla(\bar{u} - v) \cdot \nabla w \, dx + \int_\Omega g(\bar{u},v)\, w \, dx = 0, \quad \forall w \in H_0^1(\Omega), \tag{31}$$

where

$$g(\bar{u},v) = \mathbf{b} \cdot \nabla \bar{u} + c\bar{u} + \nabla \cdot (\mathbf{b} v) - cv - f \quad \in L_2(\Omega). \tag{32}$$

Plugging (30) into (31), we obtain the integral identity

$$\int_\Omega \mathbf{y}^\star \cdot \nabla w \, dx + \int_\Omega g(\bar{u},v)\, w \, dx = 0, \quad \forall w \in H_0^1(\Omega), \tag{33}$$

which shows that $\mathbf{y}^\star \in H(div,\Omega)$ and its divergence is implicitly defined as

$$\nabla \cdot \mathbf{y}^\star = g(\bar{u},v). \tag{34}$$

Hence, the integral I_1 vanishes for the above choice of v and $\mathbf{y}^\star$

$$\begin{aligned} I_1 &= \int_\Omega [f - \mathbf{b} \cdot \nabla \bar{u} - c\bar{u} + \nabla \cdot (\mathbf{y}^\star - \mathbf{b} v) + cv]\,(u - \bar{u})\; dx \\ &= \int_\Omega [\nabla \cdot \mathbf{y}^\star - g(\bar{u},v)]\,(u - \bar{u})\; dx = 0. \end{aligned} \tag{35}$$

Moreover, definition (30) renders the integral I_2 equal to zero

$$I_2 = \int_\Omega [\mathbf{y}^\star - \varepsilon \nabla(\bar{u} - v)] \cdot \nabla(u - \bar{u})\; dx = 0. \tag{36}$$

It follows from (9) that the energy norm of the error is given by

$$a(u - \bar{u}, u - \bar{u}) = R(v,\bar{u}). \tag{37}$$

In view of (30) and (34), the contributions of I_1 and I_2 to **EST** vanish as well, i.e., $\Lambda(v,\mathbf{y}^\star,\bar{u}) = 0$ and the parameter $\sigma > 0$ can be chosen arbitrarily. Thus, we have **EST** $= R(v,\bar{u})$ which equals the right-hand side of (37), i.e.,

$$|||e|||_\Omega^2 = a(u - \bar{u}, u - \bar{u}) = \underline{\mathbf{EST}}. \tag{38}$$

This proves that the upper bound **EST** is optimal and cannot be improved.

5 Practical implementation

In practice, the optimal values of v and $\mathbf{y}^\star$ are not available but usable approximations thereof can be obtained by solving the adjoint problem numerically. In the finite element framework, the discrete counterpart of (27) reads

$$a^*(v_h, w_h) = R(w_h, \bar{u}), \qquad \forall w_h \in V_h^*, \tag{39}$$

where V_h^* is a finite-dimensional subspace of $H_0^1(\Omega)$. Thus, it is natural to consider $\bar{v} := v_h \in V_h^*$ but any other approximation of v_{opt} is also admissible.

Ideally, the concomitant function $\bar{\mathbf{y}}^\star \in H(div, \Omega)$ should be chosen so as to minimize the functional $\Lambda(\bar{v}, \mathbf{y}^\star, \bar{u})$ which was shown to vanish for the optimal choice of v and $\mathbf{y}^\star$. The square of $\Lambda(\bar{v}, \mathbf{y}^\star, \bar{u})$ as defined in (18) can be estimated using the inequality $(p+q)^2 \le (1+\beta)p^2 + \left(1 + \frac{1}{\beta}\right) q^2$, $\forall \beta > 0$ which yields

$$\begin{aligned} [\Lambda(\bar{v}, \mathbf{y}^\star, \bar{u})]^2 \;\le\;& (1+\beta) C_\Omega^2 \| f - \mathbf{b} \cdot \nabla \bar{u} - c\bar{u} + \nabla \cdot (\mathbf{y}^\star - \mathbf{b}\bar{v}) + c\bar{v} \|_{0,\Omega}^2 \\ +\;& \left(1 + \frac{1}{\beta}\right) \| \mathbf{y}^\star - \varepsilon \nabla (\bar{u} - \bar{v}) \|_{0,\Omega}^2 = \eta(\bar{v}, \mathbf{y}^\star, \bar{u}, \beta). \end{aligned} \tag{40}$$

For the practical computation of the Friedrichs constant C_Ω we refer to [5, 8]. Given $\bar{v} \in H_0^1(\Omega)$ and $\beta > 0$, it is possible to determine $\bar{\mathbf{y}}^\star$ by solving a minimization problem for the quadratic functional $\eta(\bar{v}, \mathbf{y}^\star, \bar{u}, \beta)$, as explained in [5, 9]. As soon as $\bar{v}$ and $\bar{\mathbf{y}}^\star$ are available, the remaining free parameter σ can be adjusted so as to minimize the upper bound **EST** subject to (24).

A simpler way to estimate $\bar{\mathbf{y}}^\star$ for a given $\bar{v}$ is to use definition (30) and a suitable gradient averaging technique such as the standard L_2-projection

$$\int_\Omega \bar{\mathbf{y}}^\star w \, dx = \int_\Omega \varepsilon \nabla (\bar{u} - \bar{v}) w \, dx, \qquad \forall w \in V_h^\star. \tag{41}$$

It is worth mentioning that if $\bar{u} = u_h \in V_h \subset H_0^1(\Omega)$ is a true Galerkin solution of the primal problem, then $R(w_h, \bar{u}) = 0$, $\forall w_h \in V_h$. In particular, the term $I_3 = R(\bar{v}, \bar{u})$ is equal to zero if $\bar{v} \in V_h$. Likewise, if $V_h^* = V_h$ in (39), then the right-hand side vanishes and the solution is trivial: $\bar{v} = 0$, $\bar{\mathbf{y}}^\star = G_h(\varepsilon \nabla \bar{u})$, where G_h denotes the gradient averaging operator. In order to obtain a better error estimate, the adjoint problem should be solved on a finer/adapted mesh.

Another important issue is the local error control and mesh adaptivity on the basis of the proposed error estimate. In order to assess the local mesh quality, it is necessary to identify individual element contributions to the global bound given by (23). Setting $\sigma = \frac{\alpha}{\Lambda(\bar{v}, \bar{\mathbf{y}}^\star, \bar{u})}$, where $0 < \alpha < 2\varepsilon$, we obtain

$$\mathbf{EST} = \frac{1}{1 - \frac{\alpha}{2\varepsilon}} \left[R(\bar{v}, \bar{u}) + \frac{1}{2\alpha} [\Lambda(\bar{v}, \mathbf{y}^\star, \bar{u})]^2 \right] \tag{42}$$

and invoke (40) to estimate $[\Lambda(\bar{v}, \mathbf{y}^\star, \bar{u})]^2$ in terms of the functional $\eta(\bar{v}, \mathbf{y}^\star, \bar{u}, \beta)$.

Given a triangulation $\mathcal{T}_h$ of the domain Ω, the resulting upper bound $\overline{\mathbf{EST}}$ admits the following decomposition into a sum of element contributions

$$\overline{\mathbf{EST}} = \frac{1}{1-\frac{\alpha}{2\varepsilon}}\left[\sum_{K\in\mathcal{T}_h} R(\bar{v},\bar{u})|_K + \frac{1}{2\alpha}\sum_{K\in\mathcal{T}_h}\eta(\bar{v},\bar{\mathbf{y}}^\star,\bar{u},\beta)|_K\right]. \qquad (43)$$

An adaptive mesh for the primal and/or adjoint problem can be constructed using the principle of error equidistribution. An element $K\in\mathcal{T}_h$ in which (the absolute value of) $R(\bar{v},\bar{u})|_K$ and/or $\eta(\bar{v},\bar{\mathbf{y}}^\star,\bar{u},\beta)|_K$ is much greater/smaller than the average value calls for refinement/coarsening, respectively.

6 Numerical experiments

For testing purposes, we consider the 1D convection-diffusion equation

$$-\varepsilon u_{xx} + bu_x = 1 \quad \text{in} \quad \Omega = (0,1), \qquad u(0)=0, \quad u(1)=0.$$

The exact solution of this problem is given by (see, e.g., [3, Chapter 2])

$$u(x) = \frac{1}{b}\left(x - \frac{1-\exp(bx/\varepsilon)}{1-\exp(b/\varepsilon)}\right).$$

A series of experiments is performed to analyze the accuracy of the numerical solutions $\bar{u}_1$ and $\bar{u}_2$ computed using the P_1 Galerkin approximation (Test 1) and upwind finite differences (Test 2), respectively. In either case, three different meshes and three different values of the diffusion coefficient ε are considered, whereas the velocity $b=1$ remains unchanged. The corresponding mesh Peclet number is defined as $Pe_h = \frac{bh}{2\varepsilon}$, where h denotes the mesh size.

In the tables below, N_h stands for the number of elements for the primal mesh with spacing $h = 1/N_h$. The error estimate $\mathbf{EST}^{(1)}$ corresponds to $\bar{v} = 0$, while $\mathbf{EST}^{(2)}$ and $\mathbf{EST}^{(3)}$ were obtained using the approximate solutions $\bar{v}$ of the adjoint problem computed on a finer mesh with $h/4$ and $h/16$, respectively. The computation of $\bar{\mathbf{y}}^\star$ and β was performed on the finest mesh using the finite element method to minimize the functional $\eta(\bar{v},\bar{\mathbf{y}}^\star,\bar{u},\beta)$ for fixed β.

It can readily be seen that the Galerkin method (Table 1) produces more accurate results than the first-order accurate upwind difference scheme (Table 2) as long as the Peclet number is sufficiently small. As the diffusion coefficient ε decreases, the overall performance of the upwind method turns out to be better since the Galerkin solution $\bar{u}_1$ is corrupted by spurious oscillations. The best error estimates are obtained using $\mathbf{EST}^{(3)}$, which confirms that our estimate becomes sharper as the optimal values of $\bar{v}$ and $\bar{\mathbf{y}}^\star$ are approached.

Table 1: Error estimates for the P_1 Galerkin FEM approximation (Test 1).

ε	N_h	Pe_h	$\mathbf{EST}^{(1)}$	$\mathbf{EST}^{(2)}$	$\mathbf{EST}^{(3)}$	$\lVert u-\bar{u}\rVert_\Omega^2$
1.0	10	0.05	0.0021468157	0.0014326649	0.0010799035	0.0009008019
-	50	0.01	0.0001722407	0.0000747287	0.0000481127	0.0000360645
-	100	0.005	0.0000614897	0.0000219258	0.0000128429	0.0000090164
0.1	10	0.5	0.0888397973	0.0637761467	0.0473526679	0.0393769137
-	50	0.1	0.0084014376	0.0035716331	0.0022266005	0.0016629356
-	100	0.05	0.0031126097	0.0010484205	0.0005912496	0.0004164614
0.01	10	5.0	9.0524188397	2.5550768867	1.2570124355	0.6784573770
-	50	1.0	1.0829108208	0.4442401757	0.2374002674	0.1353352832
-	100	0.5	0.4930939764	0.1717207600	0.0798119754	0.0393744592

Table 2: Error estimates for the upwind difference approximation (Test 2).

ε	N_h	Pe_h	$\mathbf{EST}^{(1)}$	$\mathbf{EST}^{(2)}$	$\mathbf{EST}^{(3)}$	$\lVert u-\bar{u}\rVert_\Omega^2$
1.0	10	0.05	0.0015231711	0.0013563651	0.0011500432	0.0010760040
-	50	0.01	0.0000647638	0.0000547471	0.0000467697	0.0000439173
-	100	0.005	0.0000163155	0.0000137032	0.0000117162	0.0000110061
0.1	10	0.5	0.1010200881	0.0747522605	0.0623454955	0.0586377335
-	50	0.1	0.0062249147	0.0043659327	0.0038415249	0.0036704573
-	100	0.05	0.0016571967	0.0011559753	0.0010208106	0.0009765506
0.01	10	5.0	3.2940241077	1.7198159312	0.6979600529	0.4015226557
-	50	1.0	0.4785084381	0.1864664198	0.1551755090	0.1463239665
-	100	0.5	0.1769118447	0.0822637999	0.0638939282	0.0587330069

Figures 1 and 2 display the exact and numerical solutions as well as the distribution of element contributions to $\lVert e\rVert_\Omega^2$ and η for $\varepsilon = 0.1$. In this example, we consider $\bar{v} = 0$ so that $R(\bar{v}, \bar{u}) = 0$. This is why both the Galerkin method (Fig. 1) and the upwind scheme (Fig. 2) give rise to element contributions $\eta|_K$ which provide a reasonable estimate of the error distribution. On the other hand, the residual $R(\bar{v}, \bar{u})$ approaches $\lVert e\rVert_\Omega^2$ as $\bar{v} \to v_{opt}$ and $\bar{\mathbf{y}}^\star \to \mathbf{y}^\star_{opt}$. Therefore, the local error will be dominated by $R|_K$ rather than $\eta|_K$ if $\bar{v}$ is constructed by solving the adjoint problem on a sufficiently fine mesh.

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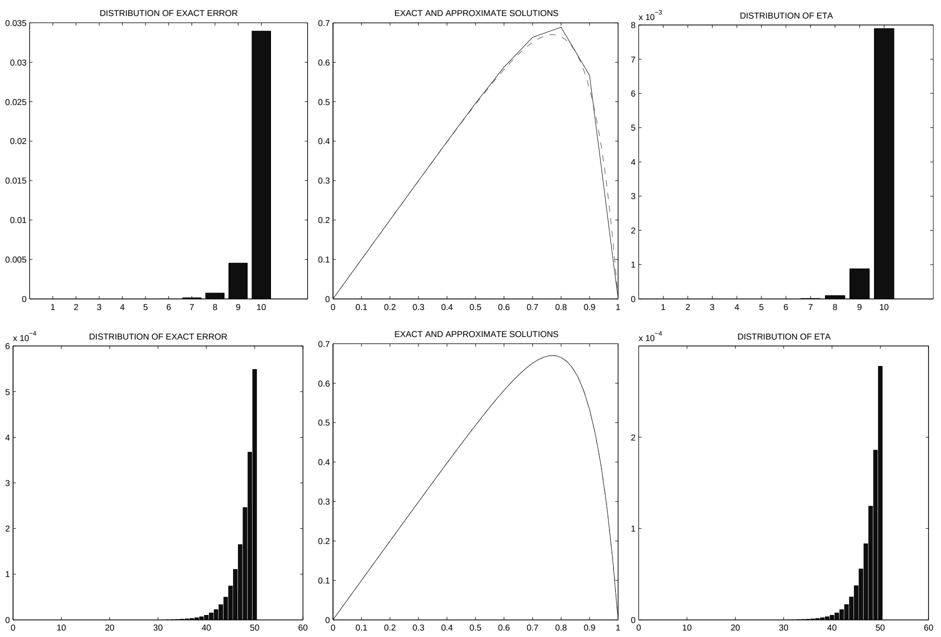

Figure 1: Galerkin method: $Pe_h = 0.5$ (top) and $Pe_h = 0.1$ (bottom).

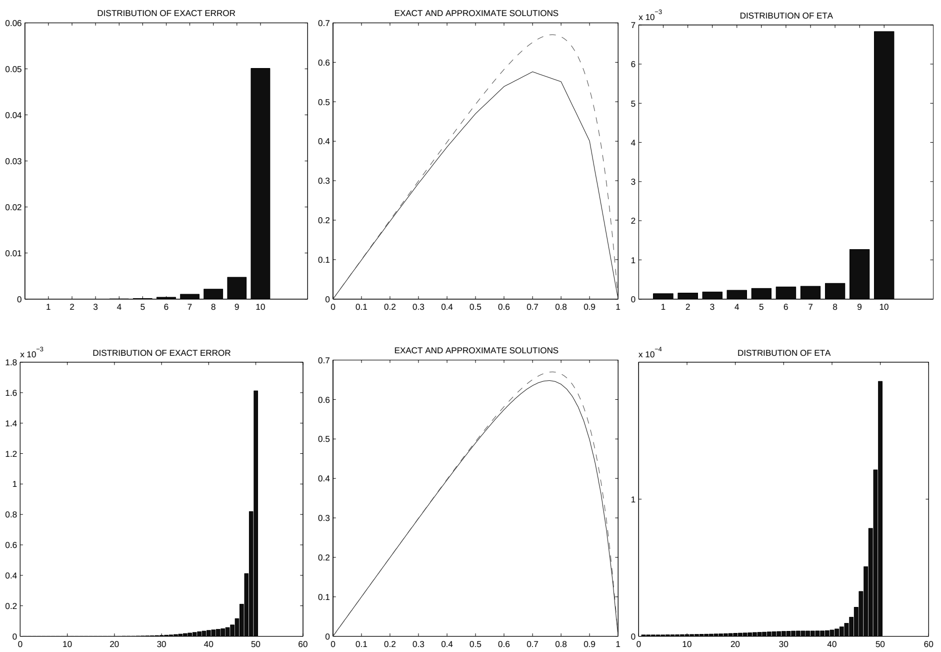

Figure 2: Upwind method: $Pe_h = 0.5$ (top) and $Pe_h = 0.1$ (bottom).

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