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Jan Brandts Sergey Korotov Michal Křížek Jakub Šolc

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
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Abstract: *This paper surveys some interesting results on nonobtuse simplices. In particular, we recall path-simplices that generalize right triangles into higher dimensions. We also deal with partitions containing only acute or only nonobtuse simplices. Such partitions are relevant in piecewise polynomial approximation theory in general, and thus also in the finite element method. Finally, we show some applications of nonobtuse simplices in algebra, mathematical analysis, graph theory, in generating geodetical meshes, etc.*

AMS subject classifications: 51M20, 65N30

Keywords: ortho-simplices, path-simplices, Delaunay triangulation, Kuhn partition, polytope.

Correspondence

Jan Brandts,
Korteweg-de Vries Institute, Faculty of Science, University of Amsterdam,
Plantage Muidergracht 24, 1018 TV Amsterdam, Netherlands,
e-mail: `brandts@science.uva.nl`

Sergey Korotov
Institute of Mathematics, Helsinki University of Technology
P. O. Box 1100, FIN–02015 Espoo, Finland,
e-mail: `sergey.korotov@hut.fi`

Michal Křížek and Jakub Šolc,
Mathematical Institute, Academy of Sciences
Žitná 25, CZ-115 67 Prague 1, Czech Republic,
e-mail: `krizek@math.cas.cz, solc@math.cas.cz`

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1. Introduction

A *simplex (d-simplex)* in the Euclidean space $\mathbb{R}^d$, $d \in \{1, 2, 3, \dots\}$, is a convex hull of $d+1$ points, all of which do not belong to the same hyperplane. Those points are said to be the *vertices* of the simplex. Opposite each vertex lies a $(d-1)$-dimensional *facet.* For $d > 1$ the *dihedral angle* α between two facets is defined by means of the inner product of their outward unit normals n_1 and n_2,

$$\cos\alpha = -n_1 \cdot n_2.$$

If all dihedral angles of a given simplex are less than 90° (less than or equal to 90°) we say that the simplex is *acute* (*nonobtuse*). Each simplex in $\mathbb{R}^d$ has $\binom{d+1}{2}$ dihedral angles.

Let $\overline{\Omega} \subset \mathbb{R}^d$ be a closed domain (i.e., the closure of a domain). If the boundary $\partial\overline{\Omega}$ is contained in a finite number of $(d-1)$-dimensional hyperplanes, we say that $\overline{\Omega}$ is *polytopic.* Moreover, if $\overline{\Omega}$ is bounded, it is called a *polytope*; in particular, $\overline{\Omega}$ is called a *polygon* for $d = 2$ and a *polyhedron* for $d = 3$.

Definition 1. By a *partition (or triangulation)* of a closed polytopic domain $\overline{\Omega}$ we mean a set of simplices whose union is $\overline{\Omega}$, any two simplices have disjoint interiors and any facet of any simplex is a facet of another simplex from the partition or belongs to the boundary $\partial\overline{\Omega}$. Moreover, we assume that the set of all simplices from the partition is countable without accumulation points in $\mathbb{R}^d$. A partition is called *acute* (*nonobtuse*)*,* if all its elements are acute (nonobtuse).

2. Acute partitions

First we present an important characterization of acute simplices (see, e.g., [18], [20, p. 110], [8]):

Theorem 1. *If $d > 2$ then each facet of an acute d-simplex is an acute $(d-1)$-simplex.*

The converse implication does not hold. For instance the tetrahedron with vertices $A = (-1, 0, 0)$, $B = (1, 0, 0)$, $C = (0, -1, \frac{1}{2})$, and $D = (0, 1, \frac{1}{2})$ has congruent acute triangular faces, but the dihedral angles at the edges AB and CD are obtuse.

Figure 1 shows a triangulation of an obtuse triangle (see [44]) and a square (see [22]) into 7 and 8 acute triangles, respectively. In 1964 Lindgren (see [41]) showed that these numbers are optimal, i.e., that they cannot be reduced. Later, Cassidy and Lord (see [10]) proved that for any $n \geq 10$ there exists a triangulation of a square into n acute triangles, whereas for $n = 9$ such a triangulation does not exist.

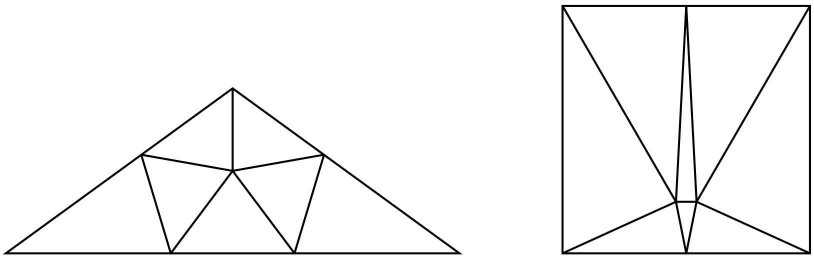


Fig. 1. Partition of an obtuse triangle and square into acute triangles.

Each triangle and also quadrangle is a plane-filler. Using this fact we can easily construct acute "periodic" triangulations of the plane. Dividing the Penrose rhombic tiles (see [47]) into 2 isosceles triangles, we can generate acute "aperiodic" triangulations with maximal angle 72°. Paper [23] presents an algorithm that enables us to decompose special polygons into "almost equilateral triangles" with the maximal angle 72°.

It is obvious that in acute (nonobtuse) triangulations of planar domains, each inner vertex is surrounded by at least five (four) triangles (cf. Fig. 1). The ratio between the corresponding numbers in $\mathbb{R}^3$ is much bigger. In [37, p. 165] it was proved that:

Theorem 2. *In any acute and nonobtuse partition of a polyhedral domain in $\mathbb{R}^3$ each inner vertex is surrounded by at least twenty and eight tetrahedra, respectively.*

These numbers are attainable, because the regular icosahedron and octahedron (see Fig. 2) can be divided into 20 acute and 8 nonobtuse tetrahedra, respectively. Their common vertex is the center of gravity in both cases. For $d = 4$, the above numbers seem to be 600:16 (cf. Conjecture 5). In fact, around 1852, Ludwig Schläfli (see [49]) studied regular polytopes in $\mathbb{R}^4$, in particular the regular 600-cell and 16-cell (also called 4-orthoplex). Their three-dimensional surfaces are formed by regular tetrahedra (see, e.g., [50]) whose convex hulls with the center of gravity G of the regular polytope form 600 acute and 16 nonobtuse 4-simplices surrounding G. For $d \geq 5$ the situation is different, due to Theorem 5 below.

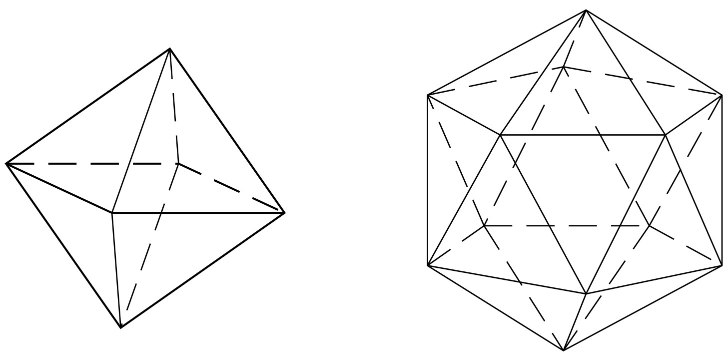


Fig. 2. The regular octahedron and icosahedron.

Generating acute partitions in $\mathbb{R}^3$ is much harder than in $\mathbb{R}^2$. For instance, it is not even known whether a cube can be decomposed into acute tetrahedra.

Aristotle in his treatise *On the Heaven* (350 BC) incorrectly conjectured that the regular tetrahedron is a space-filler (see [1, Vol. 3, Chapt. 8]). This would require the dihedral angle between its faces to be equal to 72°. Since Aristotle was a recognized person, nobody doubted his statement. Only in the Middle Ages it was realized that he was mistaken (see Fig. 3). All dihedral angles of the regular tetrahedron are equal to $\arccos\frac{1}{3}$ which, rounded to entire degrees, gives 71°. Also, Averroes (1126–1198) calculated (see [51, p. 127]) that the length of each edge of the regular icosahedron, inscribed to the unit ball, is

$$\frac{1}{5}\sqrt{10(5-\sqrt{5})} \doteq 1.05,$$

which is not 1 as it would follow from the Aristotle conjecture.

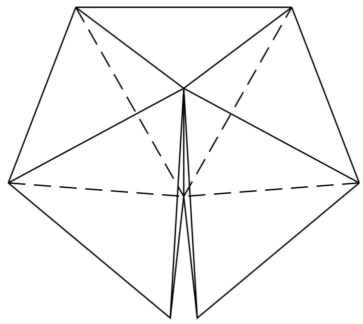


Fig. 3. The regular tetrahedron is not a space-filler. To a given face of the regular tetrahedron we may join face-to-face another regular tetrahedron in a unique way. Repeating this process, 5 regular tetrahedra may surround a common edge, but a small gap will appear, since all dihedral angles are approximately 71° only.

An algorithm for partitioning $\mathbb{R}^3$ into acute tetrahedra was given only very recently. In 2004 Eppstein, Sullivan and Üngör [15] published the following theorem.

Theorem 3. *There exists an acute partition of* $\mathbb{R}^3$.

Their elegant constructive proof of this theorem is based on the fact that the regular icosahedron (see Fig. 2) can be decomposed into 20 acute tetrahedra whose common vertex is the center of gravity of the icosahedron. Notice, moreover, that the projection of the regular icosahedron onto the plane on which it stands (on one triangle), is the regular hexagon which is a plane-filler (see Fig. 4). Congruent regular icosahedra thus may be placed into a spatial regular lattice so that two neighboring isocahedra have a common edge, but not a face. The remaining gaps can be partitioned by four different types of acute tetrahedra (see Fig. 4).

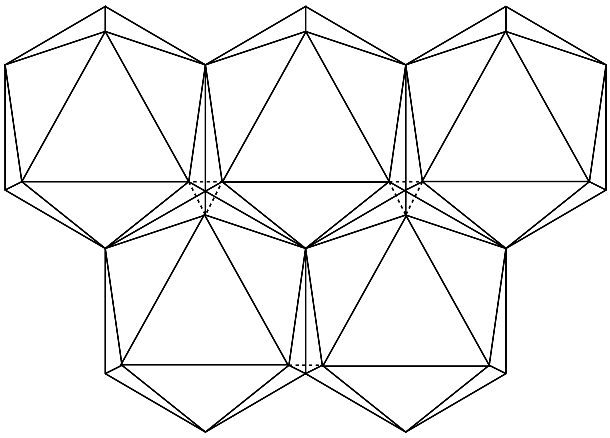


Fig. 4. Construction of an acute partition of $\mathbb{R}^3$ by means of the regular icosahedra.

Paper [15] introduces four more algorithms for generating acute partitions of $\mathbb{R}^3$ which use information about the position of atoms in crystals of zeolites. The volume of these minerals increases with pressure due to a special dense arrangement of atoms, e.g., in a silicon dioxide. Their chemical structure was studied already in 1958 (see [17]). Half a century later it was found that they could be used in the construction of acute partitions of $\mathbb{R}^3$.

The main idea is the following: Denote the centers of the particular atoms by A_1, $A_2, \ldots$. For each $i \in \{1, 2, \ldots\}$ define the corresponding *Voronoi cells* in $\mathbb{R}^d$. Properties of these convex polytopes were studied by Georgij Voronoi (1868–1908), e.g., in [56], but they were previously already defined by P. G. L. Dirichlet (1805–1859), see Fig. 5, as

$$V_i = \{x \in \mathbb{R}^d \mid \|x - A_i\| \leq \|x - A_j\| \text{ for all } j = 1, 2, \ldots\},$$

where $\|\cdot\|$ stands for the Euclidean norm. The set V_i thus contains all points $x \in \mathbb{R}^d$ whose distance from A_i is less than or equal to the distances to each of the other points A_j. Then one can show that the associated dual Delaunay triangulation (its definition is given below) is the required acute triangulation, whose vertices form the set $\{A_1, A_2, \ldots\}$ and each edge is surrounded by 5 or 6 tetrahedra.

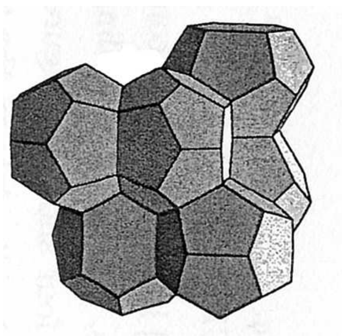


Fig. 5. Voronoi cells in $\mathbb{R}^3$ corresponding to atoms of special chemical compounds.

Definition 2. Let $D \subset \mathbb{R}^d$ be the set of all vertices of all simplices from a given triangulation. If the interiors of the circumscribed balls about any simplex from the triangulation do not contain points from D (see Fig. 6), then the triangulation is said to be *Delaunay.*

In the pioneering paper *Sur la sphère vide* (see [14]) by Boris Nikolajevič Delone[1] it was shown that for n points, which do not all belong to the same hyperplane, such a triangulation always exists. Moreover, if no $d+2$ points from D lie on the surface of a d-dimensional ball, the Delaunay triangulation is determined uniquely. Note that there are also other constructive definitions of Delaunay triangulations, e.g., in [45], [46], [52].

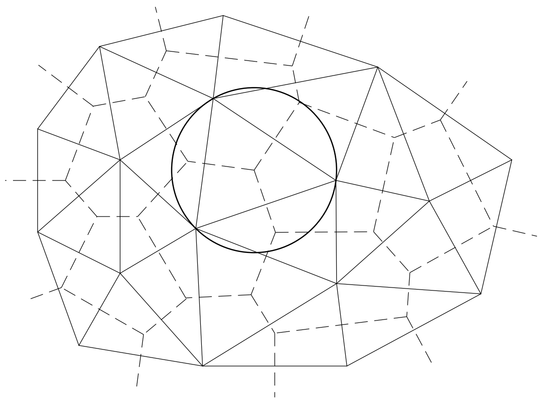


Fig. 6. Voronoi cells in $\mathbb{R}^2$ are indicated by broken lines. Notice that the vertices of the dual Delaunay triangulation are not contained in the interiors of the circumscribed balls about the particular triangles. This triangulation maximizes the minimal angle among all triangulations having the same vertices for $d = 2$.

It should be noted that one of the four above-mentioned algorithms, which uses the crystal lattice of zeolites, has the maximal dihedral angle not greater than 74.2° (see [13]). It is not known whether there exists an acute partition of $\mathbb{R}^3$ with a smaller value of the maximal dihedral angle. This angle cannot be less than 72°. This is due to the following theorem (see [35]).

Theorem 4. *In any tetrahedral partition of* $\mathbb{R}^3$ *there exist an edge, which is surrounded by at least 6 tetrahedra, and an edge, which is surrounded by at most 5 tetrahedra.*

Corollary. *In any tetrahedral partition of* $\mathbb{R}^3$ *there exist a dihedral angle less than or equal to* 60° *and a dihedral angle greater than or equal to* 72°.

From this we again see why the Aristotle conjecture cannot hold. In [13] an algorithm is given, which enables us to decompose an infinite slab of constant thickness into acute tetrahedra with maximal angle not greater than 87.7°. To divide an arbitrary tetrahedron into acute tetrahedra is still an open problem. It is also not known how to refine acute partitions in $\mathbb{R}^3$ to keep all dihedral angles acute. The main obstacle here is the fact that we

[1]His surname sounds French and therefore it is usually spelt Delaunay.

cannot use bisection of an edge, since then at least one the new angles would be not acute.

Let us point out that the dihedral angle of the regular simplex in $\mathbb{R}^d$ is

$$\alpha(d) = \arccos\frac{1}{d} < 90^\circ.$$

For $d \geq 4$ this angle is greater than 72°, since $\alpha(4) \approx 76^\circ$ and the sequence $\{\alpha(d)\}_{d=2}^{\infty}$ is increasing. Consequently, $k\alpha(d) \neq 360^\circ$ for any integer k and $d > 2$. This means that the regular simplex is not a space-filler of higher dimensional spaces. However, a more surprising assertion holds.

Theorem 5. *For $d \geq 5$ there is no acute partition of $\mathbb{R}^d$.*

Its inductive proof (see [36]) resembles Fermat's method of infinite descent. It uses Euler-Poincaré formula [43] which implies that in five-dimensional space a point cannot be surrounded by acute simplices. On the other hand, in $\mathbb{R}^4$ a point can be surrounded by at least 600 acute simplices due to the existence of the 600-cell. In spite of that we believe that the following conjecture is true.

Conjecture 1. *There is no acute partition of $\mathbb{R}^4$.*

3. Path-simplices and nonobtuse partitions

We start with a useful characterization of nonobtuse simplices (see, e.g., [8]):

Theorem 6. *Let $d > 2$. If a d-simplex is nonobtuse, then each of its facets is a nonobtuse $(d-1)$-simplex.*

Now we introduce two notions, which are not always consistently defined in the literature (cf., e.g., [49]).

Definition 3. An *ortho-simplex* in $\mathbb{R}^d$ is a simplex having d mutually orthogonal edges. A *path-simplex* is $\mathbb{R}^d$ is an ortho-simplex whose d orthogonal edges form a path (in the sense of graph theory); in particular, for $d = 3$ we shall speak about a *path-tetrahedron.*

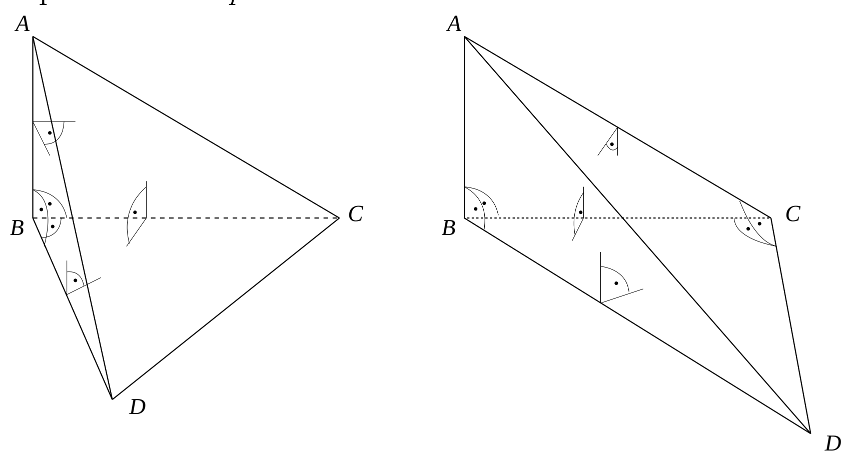


Fig. 7. Two types of ortho-simplices in $\mathbb{R}^3$. The one on the right is a path-tetrahedron whose three mutually orthogonal edges AB, BC, and CD form a path. All its faces are right triangles.

Examples. A right triangle in $\mathbb{R}^2$ is an ortho-simplex and also a path-simplex. In Fig. 7 we see two types of ortho-simplices in $\mathbb{R}^3$. The first one has three mutually orthogonal edges that share a common point B. Three of its faces are right triangles and the fourth face ACD is an acute triangle. The second type of ortho-simplex only has right triangular faces (see Fig. 7 on the right). We observe that its 3 orthogonal edges form a path. Therefore, it is a path-tetrahedron.

The following three theorems (cf. Fig. 7) can be found in the work by Fiedler [20] (see also [4], [8], [30]).

Theorem 7. *Each ortho-simplex is nonobtuse.*

Theorem 8. *Each simplex has at least d acute dihedral angles. Each ortho-simplex has exactly d acute dihedral angles.*

This theorem (see [16, p. 315 and 320]) is so nice that it was rediscovered and published fifty years later as [39], even though it can be found in Mathematical Reviews 0069507.

Theorem 9. *Let $d > 2$. A d-simplex is a path-simplex if and only if each of its facets is a path $(d-1)$-simplex.*

From this we inductively find that a d-simplex is path if and only if each of its two-dimensional faces is a right triangle (see [17], [20]).

Let us present further interesting results. We observe that the tetrahedron on the left of Fig. 7 does not contain its circumcenter. (A formula for the radius of the circumscribed ball of a simplex in $\mathbb{R}^d$ is derived in [16, p. 316].) On the other hand, the circumcenter of the tetrahedron right in Fig. 7 lies at the midpoint of the longest edge AD. This is also true in $\mathbb{R}^d$ (see [4, p. 194], [17], [20]):

Theorem 10. *An ortho-simplex contains its circumcenter if and only if it is a path-simplex.*

Notice that the circumscribed ball about a path-simplex for $d = 2$ is, in fact, the Thales circle. In 1994 Rajan proved (see [48, p. 200]) another remarkable assertion.

Theorem 11. *If each simplex in a given triangulation in $\mathbb{R}^d$ contains its circumcenter, then the triangulation is Delaunay.*

In particular, each nonobtuse triangulation in the plane is Delaunay (cf. Fig. 6). The converse implication does not hold. I.e., there exists a Delaunay triangulation in $\mathbb{R}^2$ containing obtuse angles. A combination of the previous two theorems gives the following elegant consequence.

Theorem 12. *A triangulation into path-simplices is Delaunay.*

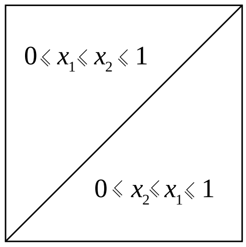
Another interesting theorem was proved by Freudenthal in his paper [21] from 1942.

Theorem 13. *The unit cube $[0, 1]^d$ can be decomposed into $d!$ path-simplices.*

Indeed, the path-simplices from the above theorem can be defined as (see Fig. 8 for $d = 2, 3$):

$$S_\sigma = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(d)} \le 1\},$$

where σ ranges over all permutation of the numbers $1, 2, \ldots, d$.



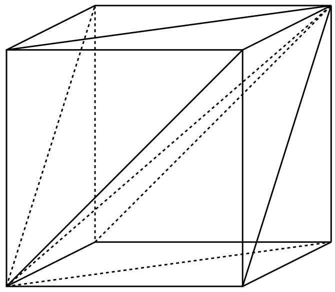


Fig. 8. Triangulation of a square and cube into 2 and 6 path-simplices, respectively.

Thus, $\mathbb{R}^d$ can be first be decomposed into d-cubes and also into path-simplices. Let us point out that the Freudenthal triangulation is also called the Kuhn partition due to the paper [38]. Another elegant idea was published by Michael Goldberg in [24]. He divided the space $\mathbb{R}^3$ into congruent infinite prisms having an equilateral triangle as cross-section. Then each prism was decomposed into congruent simplices (see Fig. 9), which can be chosen nonobtuse. If the division of each triangular prism is a mirror image of an adjacent prism, we get a nonobtuse partition of the three-dimensional space in the sense of Definition 1.

A special tetrahedral space-filler was found in 1923 by Sommerville (see [24]). The length of its two opposite edges is 2 and the other four edges have length $\sqrt{3}$. Dihedral angles at the two longer edges are right, whereas the other four dihedral angles equal to 60°. This tetrahedron can be easily decomposed into 8 congruent tetrahedra that are similar to the original one. It can also be partitioned into four path-tetrahedra.

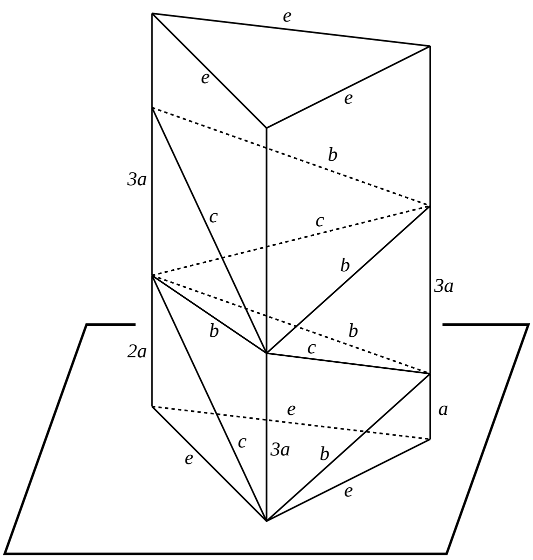


Fig. 9. Goldberg's division of an infinite triangular prism into congruent tetrahedra. If $\sqrt{2}b \geq c$, then no dihedral angle exceeds 90° (see [24, p. 353]).

Theorem 14. *Each path-simplex in $\mathbb{R}^d$ can be subdivided into d smaller path-simplices.*

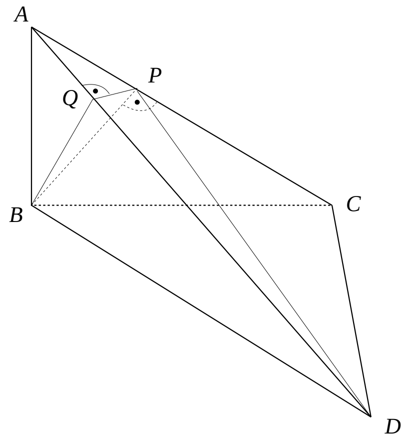


Fig. 10. Coxeter's trisection of a path-tetrahedron $ABCD$ into smaller path-tetrahedra. The points P and Q are the orthogonal projections of the points B and P on the segment AC and AD, respectively.

If $d = 2$ then any right triangle can be divided by the altitude on the hypotenuse into two smaller right triangles. Partition of a path-tetrahedron into 3 smaller path-tetrahedra (see Fig. 10) was described by Coxeter [11] in 1989. The same construction was, in fact, implicitly used by Lenhard already in 1960 to trisect tetrahedra of the so-called class $T1c$ (see [40]) to which path-simplices belong.

In [8] each path-simplex in $\mathbb{R}^d$ is decomposed into d (and also $d+1$) smaller path-simplices. The geometric interpretation of this procedure resembles the Gram-Schmidt orthogonalization process.

By Theorem 13, each d-dimensional cube can be decomposed into $d!$ path-simplices (cf. Fig. 8). However, the smallest number of simplices into which the d-dimensional cube can be decomposed is given by (see [7], [27]):

$$1, 2, 5, 16, 67, 308, 1493, \ldots$$

The four-dimensional cube can be thus divided into 16 simplices and this number cannot be reduced (see [26]). In Fig. 11 we see 5 tetrahedra that form a cube. One of them is the regular tetrahedron. The other four are ortho-simplices, but not path-tetrahedra, since they do not contain the center of the circumscribed ball. The circumcenter lies in the centre of the cube and it is common to all 5 tetrahedra.

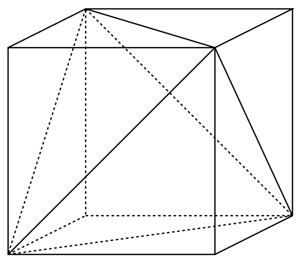


Fig. 11. Division of a cube into 5 tetrahedra.

In 1957 Hugo Hadwiger [25] stated the following conjecture:

Conjecture 2. *Each simplex in $\mathbb{R}^d$ can be subdivided into a finite number of path-simplices.*

Its validity was proven only for small values of d. If $d = 2$, then each triangle can be divided into 2 right triangles by means of the altitude to the longest edge.

In 1960 Lenhardt [40] described for $d = 3$ a method to decompose each tetrahedron into 12 path-tetrahedra. Later Böhm [6] showed that this number cannot be generally reduced.

The case $d = 4$ was solved in 1982, when A. B. Charsischwili divided a general 4-simplex into at most 730 path-simplices (see [28]). This number was reduced in 1986 by H. Kaiser to 610 path-simplices (see [29]) and in 1993 by Katrin Tschirpke to 500 path-simplices (see [53]). However, the smallest possible number is yet unknown.

K. Tschirpke also investigated the case $d = 5$. In [55] (based on her dissertation [54]) she showed that it is sufficient to use 12 598 800 path-simplices.

Each nonconvex polytope can be cut into convex polytopes by a finite number of planes whose union contains $\partial\overline{\Omega}$. Each convex polytope can be easily decomposed into simplices. If the Hadwiger conjecture is valid, then each polytope can be decomposed into a finite number of path-simplices. Path-simplices in geometry of polytopes are thus basic building blocks like atoms in nature. They are even more elementary than simplices themselves.

4. Applications

Acute and nonobtuse simplices play an important role in many areas:

I. Algebra. Consider groups of symmetries of the Platonic bodies and their generators. For instance, there are 4 generators for the cube. The corresponding 4 planes of symmetries bound a path-tetrahedron which is called the *fundamental domain.* All planes of symmetry divide the Platonic bodies into path-tetrahedra (see Fig. 12).

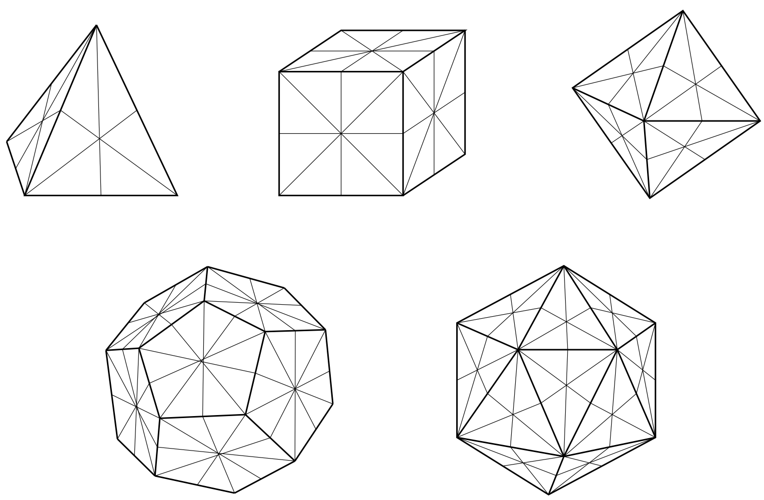


Fig. 12. Division of the Platonic bodies into path-tetrahedra.

II. Mathematical analysis. Ortho-simplices may serve as a tool for the evaluation of some integrals. For instance, by the trisection of a path-tetrahedron (see Fig. 10) into 3 smaller path-tetrahedra Coxeter proved [11] that

$$\int_1^6 \frac{\sec^{-1} x}{(x+2)\sqrt{x+1}} \left(\frac{1}{\sqrt{x+3}} + 2 \right) \mathrm{d}x = \frac{2}{15}\pi^2.$$

III. Discrete geometry. A large amount of geometric applications of nonobtuse simplices (in particular, path-tetrahedra) is given in *Geometry Junkyard* [59].

IV. Graph theory. In [20, Chapt. 14] it is shown how to use path-simplices to establish the structure of electric networks. The main theorem (based on paper [19]) solves the following problem: Which are the possible networks composed only from resistors inside a "black box", which has $n \in \{2, 3, \dots\}$ outlets.

V. Numerical mathematics. A large amount of applications of nonobtuse partitions are in numerical mathematics. Since such partitions have all dihedral angles less than or equal to 90°, the Lagrange and Hermite finite element interpolation operators have optimal approximation orders. Moreover, the standard reference simplex is an ortho-simplex. Nonobtuse and acute partitions enable us to fulfill the discrete maximum principle in solving nonlinear elliptic problems [30], [31], semiconductor equations [58], and convection-diffusion problems by means of linear finite elements. Namely, the associated stiffness matrix A is monotone (i.e., A^{-1} exists and $A^{-1} \geq 0$). The sign of particular entry of the stiffness matrix depends (see [8], [34]) on cosines of dihedral angles in the partition. Indeed, for $d \geq 2$ we have

$$(\nabla v_i)^\top \nabla v_j = -\frac{\mathrm{meas}_{d-1} F_i \ \mathrm{meas}_{d-1} F_j}{d^2 \mathrm{meas}_d^2 S} \cos \alpha_{ij}, \quad i, j = 1, \dots, d+1, \quad i \neq j,$$

where α_{ij} is the dihedral angle between faces F_i and F_j of a simplex S, v_i is a linear function that vanishes on F_i, $F_i(B_i) = 1$, and B_i is the vertex of S opposite F_i (cf. Fig. 13).

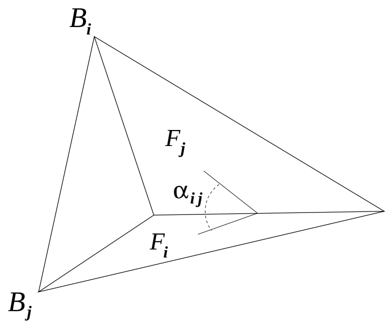


Fig. 13. Illustration of the above formula for $d = 3$.

Acute simplicial partitions are very useful in finite element analysis, since they yield irreducible and diagonally dominant stiffness matrices, when solving the equation

$$-\Delta u + bu = f$$

by linear elements in a bounded polytopic domain with given boundary conditions and $b \geq 0$ small enough. On the other hand, d-linear block elements do not guarantee, in general, the discrete maximum principle for $d \geq 2$, and for $d \geq 4$ the stiffness matrix is never irreducible and diagonally dominant (see [31]).

Such problems with $d \geq 4$ are encountered in financial mathematics and theoretical physics. In [33] the Coxeter trisection from Fig. 10 is used recursively to construct local refinements of partitions in a neighbourhood of vertices of a polyhedron by means of path-tetrahedra. Kuhn triangulation is employed for preconditioning of large problems solved by multigrid methods (see [3], [5]). In [9] it is used to prove gradient superconvergence of linear finite elements.

Nonobtuse partitions (sometimes called *of weakly acute type*) are also applied in the finite volume method [2]. For a weakened acute type condition for tetrahedral partitions see [34]. The finite volume method proposed in [16] requires a *strict Delaunay condition,* i.e., the closure of the circumscribed ball about each simplex from the partition does not contain any other simplex from the partition.

VI. Geodesy. In triangulations used in geodesy it is convenient to use acute triangles. The closer the triangle is to an equilateral triangle, the more accurately we can establish the coordinates of particular triangulation points by means of measurement of lengths of edges (and angles).

Nonobtuse simplicies are also used in mathematical genetics [42], in the Monte Carlo method for solving partial differential equations [57, p. 210], etc.

5. Open problems

Finally, we present some more conjectures to be solved.

Conjecture 3. *Each tetrahedron (or cube) can be partitioned into acute tetrahedra.*

Conjecture 4. *Each polyhedron allows a face-to-face partition into nonobtuse tetrahedra.*

Conjecture 5. *A vertex in $\mathbb{R}^4$ cannot be surrounded by less than 600 acute simplices.*

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