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A. Hannukainen S. Korotov

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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A. Hannukainen S. Korotov

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

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Abstract: *The paper is devoted to the problem of reliable control of accuracy of approximate solutions obtained in computer simulations. This task is strongly related to the so-called a posteriori error estimates, giving computable bounds for computational errors and detecting zones in the solution domain, where such errors are too large and certain mesh refinements should be performed. Mathematical model described by a linear elliptic equation with mixed boundary conditions is considered. We derive in a simple way two-sided (upper and lower) easily computable estimates for global (in terms of the energy norm) and local (in terms of linear functionals with local supports) control of the computational error understood as the deviation between the exact solution of the model and the approximation. Such two-sided estimates are completely independent of the numerical technique used to obtain approximations and can be made as close to the true errors as resources of a concrete computer used for computations allow. Main issues of practical realization of the estimation procedures proposed are discussed and several numerical tests are presented.*

AMS subject classifications: 65N15, 65N30

Keywords: a posteriori error estimation, error control in energy norm, error control in terms of linear functionals, differential equation of elliptic type, mixed boundary conditions, gradient averaging.

Correspondence

antti.hannukainen@hut.fi, sergey.korotov@hut.fi

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

Many physical and mechanical phenomena can be described by means of mathematical models presenting boundary value problems of elliptic type [6, 15, 16]. Various numerical techniques (the finite difference method (FEM), the finite element method, the finite element volume, etc) are well developed for finding approximate solutions for such problems, see, e.g., [10]. However, in order to be practically meaningful, computer simulations always require an accuracy verification of computed approximations. Such a verification is the main purpose of a posteriori error estimation methods.

In the present paper, we recall first two different ways of measuring the computational error, which is understood as the deviation $u - \bar{u}$ between the exact solution u and approximation $\bar{u}$, in the global (energy) norm and in terms of linear bounded functionals. These two ways of measurement (and also control – via a posteriori error estimation procedures) of the error are very natural and commonly used nowadays in both mathematical and engineering communities. The global error estimation normally gives a general presentation on the quality of approximation and a stopping criterion to terminate the calculations [1, 2, 3, 4, 17, 21, 28, 29, 30]. However, practitioners are often interested not only in the value of the overall error, but also in errors over certain critical (and usually local) parts of the solution domain (for example, in fracture mechanics – see [25, 26, 27] and references therein). This reason initiated another trend in a posteriori error estimation which is based on the concept of control of the computational error locally. One common way to perform such a control is to introduce a suitable linear functional ℓ related to subdomain of interest and to construct a posteriori computable estimate for $\ell(u - \bar{u})$, see [4, 5, 11, 13, 14, 22].

It is worth to mention here that most of estimates proposed so far strongly rely on the fact that the computed solutions are true finite element (FE) approximations which, in fact, rarely happens in real computations, e.g., due to quadrature rules, forcibly stopped iterative processes, various round-off errors, or even possible bugs in FE codes.

In this work, on the base of a model elliptic problem with mixed boundary conditions, we present and test numerically two relatively simple technologies for obtaining *computable guaranteed two-sided (upper and lower) estimates* needed for reliable control in both global (in the energy norm) and local (in terms of linear functionals) ways. The estimates derived are valid for *any conforming approximations* independently of numerical methods used to obtain them, and can be made *arbitrarily close* to the true errors. In real-life calculations this closeness only depends on resources of a concrete computer used. We shall also discuss main issues of a practical realization of the error control procedures proposed and present several numerical tests.

2 Formulation of Problem

For standard definitions of functional spaces and finite element terminology used in the paper we refer, e.g., to monographs [16] and [10], respectively.

2.1 Model problem

We introduce the model elliptic problem to be considered, which consists of the governing equation (1) and mixed (Dirichlet/Neumann) boundary conditions (2)–(3): Find a function u such that

$$-\mathrm{div}(A\nabla u) + cu = f \quad \text{in} \quad \Omega, \tag{1}$$

$$u = u_0 \quad \text{on} \quad \Gamma_D, \tag{2}$$

$$\nu^T \cdot A\nabla u = g \quad \text{on} \quad \Gamma_N, \tag{3}$$

where Ω is a bounded domain in $\mathbf{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$, such that $\overline{\partial\Omega} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\mathrm{meas}_{d-1}\,\Gamma_D > 0$, ν is the outward normal to the boundary, $f \in L_2(\Omega)$, $u_0 \in H^1(\Omega)$, $g \in L_2(\Gamma_N)$, $c \in L_\infty(\Omega)$, the matrix of coefficients A is symmetric, with entries $a_{ij} \in L_\infty(\Omega)$, $i, j = 1, ..., d$, and is such that

$$C_2|\xi|^2 \geq A(x)\xi \cdot \xi \geq C_1|\xi|^2 \qquad \forall \xi \in \mathbf{R}^d \quad \forall x \in \Omega. \tag{4}$$

In addition, let us assume that almost everywhere

$$c \geq 0 \quad \text{in} \quad \Omega, \tag{5}$$

and stand the denotation

$$\Omega^c := \mathrm{supp}\; c = \{x \in \Omega \mid c(x) > 0\}. \tag{6}$$

It is common practice to pose problem (1)–(3) in the so-called weak form: Find $u \in u_0 + H^1_{\Gamma_D}(\Omega)$ such that

$$\int_\Omega A\nabla u \cdot \nabla w\, dx + \int_\Omega cuw\, dx = \int_\Omega fw\, dx + \int_{\Gamma_N} gw\, ds \qquad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{7}$$

where

$$H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}. \tag{8}$$

If we define bilinear form $a(\cdot,\cdot)$ and linear form $F(\cdot)$ as follows

$$a(v, w) := \int_\Omega A\nabla v \cdot \nabla w\, dx + \int_\Omega cvw\, dx, \quad v, w \in H^1(\Omega), \tag{9}$$

$$F(w) := \int_\Omega fw\, dx + \int_{\Gamma_N} gw\, ds, \quad w \in H^1(\Omega), \tag{10}$$

then weak formulation (7) can be written in a short form: Find $u \in u_0 + H^1_{\Gamma_D}(\Omega)$ such that $a(u, w) = F(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega)$.

Remark 2.1 *The weak solution defined by* (7) *exists and is unique in view of well-known Lax-Milgram lemma (see, e.g.,* [10]*) for example if* $c(x) \geq c_0 > 0$ *on some ball* $B \subset \Omega$*.*

The so-called *energy functional* J of problem (7) is defined as follows

$$J(w) := \frac{1}{2}a(w,w) - F(w), \qquad w \in H^1(\Omega), \tag{11}$$

and the corresponding *energy norm* is defined as $\sqrt{a(\cdot,\cdot)}$.

Remark 2.2 *It is well-known that problem* (7) *is equivalent to the problem of finding the minimizer of the energy functional* (11) *over the space* $H^1_{\Gamma_D}(\Omega)$ *and such a minimizer is the solution of problem* (7)*.*

2.2 Types of error control

Let $\bar{u}$ be *any function* from $u_0 + H^1_{\Gamma_D}(\Omega)$ (e.g., computed by some numerical method) considered as an approximation of u. It is a natural practice to measure the overall accuracy of the approximation $\bar{u}$ in terms of the above-defined energy norm. Thus, our first goal is to construct reliable and easily computable two-sided estimates for controlling the following value

$$a(u-\bar{u}, u-\bar{u}) = \int\limits_\Omega A\nabla(u-\bar{u})\cdot\nabla(u-\bar{u})\,dx + \int\limits_\Omega c(u-\bar{u})^2\,dx. \tag{12}$$

The second type of error control considered in the paper is two-sided estimation of the value of the deviation $u-\bar{u}$ in terms of some linear bounded functional ℓ

$$\ell(u-\bar{u}). \tag{13}$$

Remark 2.3 *It is clear that existence of an estimate for* (13) *also allows to estimate of the value* $\ell(u)$ *(often called quantity of interest or goal-oriented quantity* [1]*). Really,* $\ell(u) = \ell(u-\bar{u}) + \ell(\bar{u})$ *where* $\ell(\bar{u})$ *is computable and* $\ell(u-\bar{u})$ *is estimated. The value of* $\ell(u)$ *can be sometimes more important to know than the solution* u *itself (cf.* [18, Chapt. VII] *and also* [25, 26, 27]*).*

Remark 2.4 *If the functional* ℓ *in* (13) *is defined as some integral over small subdomain (or line) in* $\overline{\Omega}$*, then reliable two-sided estimation of* $\ell(u-\bar{u})$ *helps to control the behaviour of the error* $u-\bar{u}$ *locally in that subdomain (or over the line). For example, one can be interested in estimation of* $\ell(u-\bar{u}) = \int\limits_S \varphi(u-\bar{u})\,dx$ *with* S *be a subdomain in* Ω *or a line in* Γ_N *(where the solution is also unknown).*

2.3 Inequalities and constants

In what follows we shall need the Friedrichs inequality

$$\|w\|_{0,\Omega} \leq C_{\Omega,\Gamma_D} \|\nabla w\|_{0,\Omega} \quad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{14}$$

and the inequality in the trace theorem

$$\|w\|_{0,\partial\Omega} \leq C_{\partial\Omega} \|w\|_{1,\Omega} \quad \forall w \in H^1(\Omega), \tag{15}$$

where C_{Ω,Γ_D} and $C_{\partial\Omega}$ are positive constants, depending only on Ω, Γ_D, and $\partial\Omega$. The above used denotation $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{1,\Omega}$ stand for the standard norms in $L_2(\Omega)$ and $H^1(\Omega)$, respectively. The symbol $\|\cdot\|_{0,\partial\Omega}$ means the norm in $L_2(\partial\Omega)$. Proofs of inequalities (14) and (15) can be found, e.g., in [20].

3 Two-Sided Estimates of Error in Energy Norm

In this section we shall employ the denotation χ_S for a characteristic function of set S, i.e., $\chi_S(x) = 1$ if $x \in S$, and $\chi_S(x) = 0$ if $x \notin S$, and also stand the denotation $|\!|\!|y|\!|\!|_\Omega := \sqrt{\int\limits_\Omega Ay \cdot y \, dx}$ for $y \in L_2(\Omega, \mathbf{R}^d)$.

3.1 Upper estimate

Theorem 3.1 *For the error in the energy norm* (12) *we have the following upper estimate*

$$a(u-\bar{u}, u-\bar{u}) \leq \|\frac{1}{\sqrt{c}}(f + div y^* - c\bar{u})\|^2_{0,\Omega^c} +$$
$$+(1+\alpha)|\!|\!|A^{-1}y^* - \nabla\bar{u}|\!|\!|^2_\Omega + (1+\frac{1}{\alpha})(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f + div y^*\|^2_{0,\Omega\setminus\overline{\Omega}^c} \tag{16}$$
$$+(1+\frac{1}{\alpha})(1+\frac{1}{\beta})C^2_{\Omega,\partial\Omega}\|g - \nu^T \cdot y^*\|^2_{0,\Gamma_N},$$

where α and β are arbitrary positive real numbers and y^* is any function from $H_N(\Omega, \mathrm{div}) := \{y \in L_2(\Omega, \mathbf{R}^d) \mid \mathrm{div}\, y \in L_2(\Omega),\ \nu^T \cdot y \in L_2(\Gamma_N)\}$.

P r o o f : First of all, we notice that it actually holds (cf. (6))

$$a(u-\bar{u}, u-\bar{u}) = |\!|\!|\nabla(u-\bar{u})|\!|\!|^2_\Omega + \|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}. \tag{17}$$

Further, using the fact that $u - \bar{u} \in H^1_{\Gamma_D}(\Omega)$, integral identity (7), the Green formula, and simple regroupping of terms in below we observe that

$$a(u-\bar{u}, u-\bar{u}) = \int\limits_\Omega f(u-\bar{u})dx + \int\limits_{\Gamma_N} g(u-\bar{u})\, ds - \int\limits_\Omega A\nabla\bar{u} \cdot \nabla(u-\bar{u})\, dx$$

$$
\begin{aligned}
-\int_\Omega c\bar{u}(u-\bar{u})\,dx &= \int_\Omega (f-c\bar{u})(u-\bar{u})\;dx + \int_{\Gamma_N} g(u-\bar{u})\,ds \\
&- \int_\Omega (A\nabla\bar{u}-y^*)\cdot\nabla(u-\bar{u})\;dx - \int_\Omega y^*\cdot\nabla(u-\bar{u})\;dx =
\end{aligned} \tag{18}
$$

$$
= \int_\Omega (f+\operatorname{div} y^* - c\bar{u})(u-\bar{u})\;dx - \int_\Omega A(\nabla\bar{u}-A^{-1}y^*)\cdot\nabla(u-\bar{u})\;dx
$$

$$
+ \int_{\Gamma_N} g(u-\bar{u})\,ds - \int_{\Gamma_N} \nu^T\cdot y^*(u-\bar{u})\,ds =
$$

$$
= \int_\Omega A(A^{-1}y^*-\nabla\bar{u})\cdot\nabla(u-\bar{u})\;dx + \int_\Omega (f+\operatorname{div} y^* - c\bar{u})(u-\bar{u})\;dx
$$

$$
+ \int_{\Gamma_N} (g-\nu^T\cdot y^*)(u-\bar{u})\,ds,
$$

where y^* is any function from the space $H_N(\Omega, \operatorname{div})$ defined in the conditions of the theorem.

Now, the right-hand side (RHS) of equality (18) can be estimated, using the Caushy-Schwarz inequality, denotation (6), and trace inequality (15), from above as follows

$$
\text{RHS of (18)} \le \|\!|A^{-1}y^*-\nabla\bar{u}\|\!|_\Omega\,\|\!|\nabla(u-\bar{u})\|\!|_\Omega + \|g-\nu^T\cdot y^*\|_{0,\Gamma_N}\|u-\bar{u}\|_{0,\Gamma_N}
$$

$$
+ \int_\Omega (f+\operatorname{div} y^* - c\bar{u})(u-\bar{u})\;dx \le
$$

$$
\le \|\!|A^{-1}y^*-\nabla\bar{u}\|\!|_\Omega\,\|\!|\nabla(u-\bar{u})\|\!|_\Omega + \|g-\nu^T\cdot y^*\|_{0,\Gamma_N} C_{\partial\Omega}\|u-\bar{u}\|_{1,\Omega} \tag{19}
$$

$$
+ \int_{\Omega^c} \frac{1}{\sqrt{c}}(f+\operatorname{div} y^* - c\bar{u})\,\sqrt{c}(u-\bar{u})\;dx + \int_{\Omega\setminus\overline{\Omega}^c} (f+\operatorname{div} y^* - c\bar{u})\,(u-\bar{u})\;dx.
$$

Further, using the ellipticity condition (4), Friedrichs inequality (14), and the Young inequality

$$
|a\,b| \le \frac{1}{2}a^2 + \frac{1}{2}b^2, \tag{20}
$$

we observe that

$$
\text{RHS of (19)} \le \Big(\|\!|A^{-1}y^*-\nabla\bar{u})\|\!|_\Omega + \frac{C_{\partial\Omega}\sqrt{1+C^2_{\Omega,\Gamma_D}}}{\sqrt{C_1}}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}\Big)\|\!|\nabla(u-\bar{u})\|\!|_\Omega
$$

$$+\frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}+\frac{1}{2}\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}+\int\limits_{\Omega}\chi_{\Omega\setminus\overline{\Omega}^c}(f+\operatorname{div}y^*-c\bar{u})\,(u-\bar{u})\,dx\leq$$

$$\leq\Big(\|\!|A^{-1}y^*-\nabla\bar{u})\|\!|_{\Omega}+C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}\Big)\|\!|\nabla(u-\bar{u})\|\!|_{\Omega} \qquad (21)$$

$$+\frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}+\frac{1}{2}\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}+\|\chi_{\Omega\setminus\overline{\Omega}^c}(f+\operatorname{div}y^*-c\bar{u})\|_{0,\Omega}\,\|u-\bar{u}\|_{0,\Omega},$$

where $C_{\Omega,\partial\Omega}:=\frac{C_{\partial\Omega}\sqrt{1+C^2_{\Omega,\Gamma_D}}}{\sqrt{C_1}}$.

Regrouping terms in RHS of (21) and using again the Young inequality (20), we get an estimate

$$\text{RHS of (21)}\leq\Big(\|\!|A^{-1}y^*-\nabla\bar{u})\|\!|_{\Omega}+C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}+\frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}}\|f+\operatorname{div}y^*-c\bar{u}\|_{0,\Omega\setminus\overline{\Omega}^c}\Big)\times$$

$$\times\|\!|\nabla(u-\bar{u})\|\!|_{\Omega}+\frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}+\frac{1}{2}\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}\leq$$

$$\leq\frac{1}{2}\Big(\|\!|A^{-1}y^*-\nabla\bar{u})\|\!|_{\Omega}+C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}+\frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}}\|f+\operatorname{div}y^*-c\bar{u}\|_{0,\Omega\setminus\overline{\Omega}^c}\Big)^2 \qquad (22)$$

$$+\frac{1}{2}\|\!|\nabla(u-\bar{u})\|\!|^2_{\Omega}+\frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}+\frac{1}{2}\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}.$$

Using now (17) and the final inequality resulting from (18)–(19) and (21)–(22), multiplying it by two and regroupping, we immediately get for the error in the energy norm that

$$a(u-\bar{u},u-\bar{u})=\|\!|\nabla(u-\bar{u})\|\!|^2_{\Omega}+\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}\leq\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}$$

$$+\Big(\|\!|A^{-1}y^*-\nabla\bar{u}\|\!|_{\Omega}+\frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}}\|f+\operatorname{div}y^*\|_{0,\Omega\setminus\overline{\Omega}^c}+C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}\Big)^2. \qquad (23)$$

Finally, using two times the inequality $(a+b)^2\leq(1+\lambda)a^2+(1+\frac{1}{\lambda})b^2$ $(\lambda>0)$ for the terms in the round brackets in (23), we get estimate (16). □

3.2 Lower estimate

Theorem 3.2 *For the error in the energy norm* (12) *we have the following lower bound*

$$a(u-\bar{u}, u-\bar{u}) \geq 2(J(\bar{u}) - J(w)), \tag{24}$$

where w is any function from $H^1_{\Gamma_D}(\Omega)$ and the functional J is defined in (11).

P r o o f : First, we prove that

$$a(u-\bar{u}, u-\bar{u}) = 2(J(\bar{u}) - J(u)). \tag{25}$$

Really, we have

$$2(J(\bar{u}) - J(u)) = a(\bar{u}, \bar{u}) - 2F(\bar{u}) - a(u,u) + 2F(u)$$

$$= a(\bar{u}, \bar{u}) - a(u,u) + 2F(u-\bar{u}) = a(\bar{u}, \bar{u}) - a(u,u) + 2a(u, u-\bar{u})$$

$$= a(\bar{u}, \bar{u}) + a(u,u) - 2a(u,\bar{u}) = a(u-\bar{u}, u-\bar{u}).$$

Since u minimizes the energy functional, we have $J(u) \leq J(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega)$, which proves (24). □

3.3 Comments on two-sided estimates (16) and (24)

- In order to derive the upper (16) and the lower (24) estimates, we did not specify the function $\bar{u}$ to be a finite element approximation (or computed by some another numerical method). In fact, it is simply any function from the set $u_0 + H^1_{\Gamma_D}(\Omega)$.

- The upper estimate (16) cannot be improved. Really, if one takes $y^* = A\nabla u$, which obviously belongs to $H_N(\Omega, \mathrm{div})$, then the last two terms in the right-hand side of (16) vanish. Further, taking $\alpha = 0$, we finally observe that the inequality (16) holds as equality. To prove that the lower estimate (24) cannot be improved either, we should, obviously, take $w = u \in H^1_{\Gamma_D}(\Omega)$ and use (25).

- The upper estimate (16) contains only two global constants, C_{Ω,Γ_D} and $C_{\partial\Omega}$, which do not depend on the computational process. They have to be computed (or accurately estimated from above) only once when the problem is posed.

- In many works, devoted to a posteriori error estimation, one usually takes $c \equiv 0$. In this case $a(u-\bar{u}, u-\bar{u}) = \|\nabla(u-\bar{u})\|^2_\Omega$, the set $\Omega^c = \emptyset$, and the estimate (16) takes a simpler form

$$a(u-\bar{u}, u-\bar{u}) \leq (1+\alpha)\|A^{-1}y^* - \nabla\bar{u}\|^2_\Omega + (1+\frac{1}{\alpha})(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f + \mathrm{div}\, y^*\|^2_{0,\Omega}$$

$$+(1+\frac{1}{\alpha})(1+\frac{1}{\beta})C^2_{\Omega,\partial\Omega}\|g - \nu^T \cdot y^*\|^2_{0,\Gamma_N}. \tag{26}$$

- For the pure Dirichlet boundary condition, the third term in RHS of (26) does not exist, and, since the estimate is valid for any positive β, we can take it be zero. Then, we get the estimate

$$a(u-\bar{u},u-\bar{u}) \le (1+\alpha)|||A^{-1}y^*-\nabla\bar{u}|||^2_\Omega + \\ + (1+\frac{1}{\alpha})\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f+\operatorname{div}y^*\|^2_{0,\Omega}. \tag{27}$$

- The upper estimate (27) was first obtained in [21] using the complicated tools of the duality theory, and later it was obtained in [23] for the Poisson equation, using an idea of the Helmholz decomposition of $L_2(\Omega,\mathbf{R}^d)$. The estimate (26) is derived in [24] using the duality theory again. Our approach of derivation of the estimates is different from those used in the above mentioned works and is simplier.

- In the case of pure Dirichlet condition we have to compute, or estimate from above, only one constant C_{Ω,Γ_D}.

In what follows we shall use the following denotations for the upper and lower bounds of the error in the energy norm (12)

$$M^\oplus(\bar{u},y^*,\alpha,\beta) = \|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}+ \\ +(1+\alpha)|||A^{-1}y^*-\nabla\bar{u}|||^2_\Omega + (1+\frac{1}{\alpha})(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f+\operatorname{div}y^*\|^2_{0,\Omega\setminus\overline{\Omega}^c} \\ +(1+\frac{1}{\alpha})(1+\frac{1}{\beta})C^2_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|^2_{0,\Gamma_N}, \tag{28}$$

and

$$M^\ominus(\bar{u},w) = 2(J(\bar{u})-J(w)). \tag{29}$$

Sometimes we shall use only a short denotation $M^\oplus$ or $M^\ominus$ for the bounds if it does not lead to misunderstanding.

4 Two-Sided Estimates for Local Errors

Two-sided estimates for controlling the error $u-\bar{u}$ in terms of linear functional (13) are essentially based on the usage of an auxiliary (often called *adjoint*) problem formulated below.

Adjoint Problem: Find $v \in H^1_{\Gamma_D}(\Omega)$ such that

$$\int\limits_\Omega A\nabla v\cdot\nabla w\,dx + \int\limits_\Omega cvw\,dx = \ell(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega). \tag{30}$$

The adjoint problem can be rewritten in a shorter form similarly to the main problem (7): Find $v \in H^1_{\Gamma_D}(\Omega)$ such that $a(u,w) = \ell(w) \quad \forall w \in$

$H^1_{\Gamma_D}(\Omega)$. In particular this means, that the bilinear forms of the main and adjoint problems coincide.

The adjoint problem is uniquely solvable due to the assumption that ℓ is a linear bounded functional. However, the exact solution v of it is usually very hard (or even impossible) to find in analytical form and, thus, we only have some approximation for v, which we denote by the symbol $\bar{v}$ in what follows, assuming only that $\bar{v} \in H^1_{\Gamma_D}(\Omega)$.

Theorem 4.1 *The following error decomposition holds*

$$\ell(u-\bar{u}) = E_0(\bar{u},\bar{v}) + E_1(u-\bar{u}, v-\bar{v}), \tag{31}$$

where

$$E_0(\bar{u},\bar{v}) = \int\limits_\Omega f\bar{v}\,dx + \int\limits_{\Gamma_N} g\bar{v}\,ds - \int\limits_\Omega A\nabla\bar{v}\cdot\nabla\bar{u}\,dx - \int\limits_\Omega c\bar{v}\bar{u}\,dx, \tag{32}$$

$$E_1(u-\bar{u}, v-\bar{v}) = \int\limits_\Omega A\nabla(u-\bar{u})\cdot\nabla(v-\bar{v})\,dx + \int\limits_\Omega c(u-\bar{u})(v-\bar{v})\,dx. \tag{33}$$

P r o o f : In view of integral identities (30) and (7), and using the fact that $u-\bar{u} \in H^1_{\Gamma_D}(\Omega)$, we observe that

$$\ell(u-\bar{u}) = \int\limits_\Omega \mathrm{A}\nabla v\cdot\nabla(u-\bar{u})\,dx + \int\limits_\Omega cv(u-\bar{u})\,dx$$

$$= \int\limits_\Omega \mathrm{A}\nabla(v-\bar{v})\cdot\nabla(u-\bar{u})\,dx + \int\limits_\Omega c(v-\bar{v})(u-\bar{u})\,dx + \int\limits_\Omega \mathrm{A}\nabla\bar{v}\cdot\nabla(u-\bar{u})\,dx + \int\limits_\Omega c\bar{v}(u-\bar{u})\,dx$$

$$= E_1(u-\bar{u}, v-\bar{v}) + \int\limits_\Omega \mathrm{A}\nabla\bar{v}\cdot\nabla u\,dx + \int\limits_\Omega c\bar{v}u\,dx - \int\limits_\Omega \mathrm{A}\nabla\bar{v}\cdot\nabla\bar{u}\,dx - \int\limits_\Omega c\bar{v}\bar{u}\,dx$$

$$= E_1(u-\bar{u}, v-\bar{v}) + \int\limits_\Omega f\bar{v}\,dx + \int\limits_{\Gamma_N} g\bar{v}\,ds - \int\limits_\Omega \mathrm{A}\nabla\bar{v}\cdot\nabla\bar{u}\,dx - \int\limits_\Omega c\bar{v}\bar{u}\,dx$$

$$= E_0(\bar{u},\bar{v}) + E_1(u-\bar{u}, v-\bar{v}). \qquad \square$$

The first term E_0 is, obviously, directly computable once we have $\bar{u}$ and $\bar{v}$ computed, but the term E_1 contains unknown gradients ∇u and ∇v. In order to estimate it, we notice first that $E_1(u-\bar{u}, v-\bar{v}) \equiv a(u-\bar{u}, v-\bar{v})$. Further, the following relation obviously holds for any positive α:

$$2E_1(u-\bar{u}, v-\bar{v}) = a(\alpha(u-\bar{u}) + \frac{1}{\alpha}(v-\bar{v}), \alpha(u-\bar{u}) + \frac{1}{\alpha}(v-\bar{v}))$$

$$-\alpha^2 a(u-\bar{u}, u-\bar{u}) - \frac{1}{\alpha^2}a(v-\bar{v}, v-\bar{v}). \tag{34}$$

The last two terms in the above identity present the errors in the energy norm for main and adjoint problems. Thus, we can immediately use the two-sided estimates from Section 3, written in somewhat simplified form:

$$M^{\ominus} \leq a(u-\bar{u}, u-\bar{u}) \leq M^{\oplus}, \quad M^{\ominus}_{ad} \leq a(v-\bar{v}, v-\bar{v}) \leq M^{\oplus}_{ad}, \tag{35}$$

where subindex "*ad*" means that the corresponding estimate is obtained for the adjoint problem.

As far it concerns the first term in the right-hand side of (34), we observe that

$$a(\alpha(u-\bar{u}) + \frac{1}{\alpha}(v-\bar{v}), \alpha(u-\bar{u}) + \frac{1}{\alpha}(v-\bar{v})) =$$
$$= a((\alpha u + \frac{1}{\alpha}v) - (\alpha\bar{u} + \frac{1}{\alpha}\bar{v}), (\alpha u + \frac{1}{\alpha}v) - (\alpha\bar{u} + \frac{1}{\alpha}\bar{v})). \tag{36}$$

The function $\alpha u + \frac{1}{\alpha}v$ can be perceived as the solution of the following problem (called as the *mixed problem* in what follows): Find $u_\alpha \in u_0 + H^1_{\Gamma_D}(\Omega)$ such that

$$\int\limits_\Omega A\nabla u_\alpha \cdot \nabla w \, dx + \int\limits_\Omega c u_\alpha w \, dx = \alpha F(w) + \frac{1}{\alpha}\ell(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{37}$$

which is uniquely solvable due to the fact that $\alpha F(w) + \frac{1}{\alpha}\ell(w)$ is, obviously, also linear bounded functional.

The function $\alpha\bar{u} + \frac{1}{\alpha}\bar{v} \in H^1_{\Gamma_D}(\Omega)$ can be considered as an approximation of u_α, and we can again apply the techniques of Section 3 in order to obtain the following two-sided estimates (writen again in simplified form)

$$M^{\ominus}_{mix} \leq a(\alpha(u-\bar{u}) + \frac{1}{\alpha}(v-\bar{v}), \alpha(u-\bar{u}) + \frac{1}{\alpha}(v-\bar{v})) \leq M^{\oplus}_{mix}, \tag{38}$$

where subindex "*mix*" means that the estimates are obtained for the mixed problem.

Further, we immediately observe that

$$\frac{1}{2}(M^{\ominus}_{mix} - \alpha^2 M^{\oplus} - \frac{1}{\alpha^2}M^{\oplus}_{ad}) \leq E_1(u-\bar{u}, v-\bar{v}), \tag{39}$$

and

$$E_1(u-\bar{u}, v-\bar{v}) \leq \frac{1}{2}(M^{\oplus}_{mix} - \alpha^2 M^{\ominus} - \frac{1}{\alpha^2}M^{\ominus}_{ad}). \tag{40}$$

The above considerations can be summarized as the following theorem.

Theorem 4.2 *For the error in terms of linear functional $\ell(u-\bar{u})$ we have the following upper estimate*

$$\ell(u-\bar{u}) \leq E_0(\bar{u}, \bar{v}) + \frac{1}{2}(M^{\oplus}_{mix} - \alpha^2 M^{\ominus} - \frac{1}{\alpha^2}M^{\ominus}_{ad}), \tag{41}$$

and the following lower estimate

$$\ell(u-\bar{u}) \geq E_0(\bar{u}, \bar{v}) + \frac{1}{2}(M^{\ominus}_{mix} - \alpha^2 M^{\oplus} - \frac{1}{\alpha^2}M^{\oplus}_{ad}), \tag{42}$$

where the directly computable term $E_0(\bar{u}, \bar{v})$ is defined in (32).

5 Practical Realization

5.1 Construction and optimization of two-sided estimates

We briefly consider here main issues of practical realization of the above described error estimation technologies in the framework of the most popular numerical technique – finite element method (FEM). Nevertheless, we do not use any immanent properties of finite element approximations in what follows and use the FEM terminology only for convenience.

Since estimation of the error in terms of linear functional reduced to the two-sided estimation of the error in the global energy norm for three similar problems (main, adjoint and mixed ones), we only consider in below the case of the original problem (7).

The approximation $\bar{u}$ is further assumed to be obtained by FEM, and denoted as u_h. We also suppose that computations are performed on a series of successive meshes $\mathcal{T}_{h_1}$, $\mathcal{T}_{h_2}$, $\mathcal{T}_{h_3}$, $\dots$, where $h = h_1 > h_2 > h_3 > \dots$, and, thus, we always have in hands several successive approximations u_{h_1}, u_{h_2}, u_{h_3}, $\dots$. Such a situation is quite typical in practical calculations which use modern software packages.

On computation of global constants C_{Ω,Γ_D} and $C_{\partial\Omega}$: The constant C_{Ω,Γ_D} is determined via the smallest eigenvalue λ_Ω of the Laplacian in Ω with homogeneous boundary conditions, $C_{\Omega,\Gamma_D} = \frac{1}{\sqrt{\lambda_\Omega}}$. Thus, only estimation of λ_Ω from below is needed. In the case of homogeneous Dirichlet boundary condition this task is easily solved as proposed by S. Mikhlin in [19, p. 18] by enclosing the solution domain into a rectangular parallelepiped, for which we can easily obtain the exact value of the smallest eigenvalue which is smaller than λ_Ω. Also, suitable upper estimates of C_{Ω,Γ_D} for some conical domains are presented in [6]. On the contrary, estimation of the constant $C_{\partial\Omega}$ seems to be still an open problem for a general case. However, one trick on estimation of this constant for a quite special case is proposed in [24, Remark 3.3]. More sofisticated techniques for estimation of C_{Ω,Γ_D} and $C_{\partial\Omega}$ from above, suitable for the purposes of a posteriori error analysis, and also another numerical tests, will be presented in our subsequent paper [7].

Remark 5.1 *We notice that the other existing estimation techniques (of residual-type) commonly involve many unknown constants, usually related to patches of computational meshes used. Those constants are very hard to estimate (from above) and their computation, in general, leads to a very big overestimation of the error even in simple cases (see* [9]*). Moreover, such constants have to be always recomputed if we perform adaptive computations and change the computational mesh. On the contrary, the constants* C_{Ω,Γ_D} *and* $C_{\partial\Omega}$ *remain the same under any change in meshes.*

On minimization of upper bound: A "coarse" upper bound can be immediately computed using values $y^* = G_\mu(\nabla u_\mu)$, where $\mu = h_1, h_2, h_3, \dots$, and G_μ is some commonly used *gradient averaging operator* [8, 12]. However, more sharp estimates require a real minimization of the upper bound with respect to the "free" variables y^*, α, β, which can be performed by a direct minimization of it or by finding the minimizer as a solution of the respective system of linear equations.

On computation of lower bound: The estimate (24) has a practical meaning only if it provides with a positive lower bound for the (positive) error. This can be obtained if we recall that one normally tries to have $J(u_{h_1}) > J(u_{h_2}) > J(u_{h_3}) > \dots$, in a series of succesive computations, which immediately suggests meaningfull lower bounds as follows

$$a(u - \bar{u}, u - \bar{u}) = a(u - u_h, u - u_h) \geq 2(J(u_h) - J(u_\mu)) > 0, \tag{43}$$

where $\mu = h_2, h_3, \dots$. We note that our form of the lower estimate is different from that one presented in [22].

On mesh adaptativity: Both estimates (16) and (24) have integral form, i.e., they can be represented as integrals over the solution domain Ω. This suggests a straightforward way for a mesh adaptation. Roughly speaking, we refine only those elements of the mesh whose contributions to the integrals in our two-sided estimates are too high.

5.2 On construction of error indicators

First of all, we notice that in the case when the coefficient $c \equiv 0$, the error representations (12) and (31) (or, actually, the term E_1 in (33)) contain only unknown unknown values of gradients ∇u and ∇v, and no unknown values of the functions u and v are involved at all.

This observation suggests a procedure of replacing the unknown gradients of the exact solutions of the main and adjoint problems by the corresponding averaged gradients if the approximations $\bar{u}$ and $\bar{v}$ are computed by FEM (in this case we denote them as u_h and v_τ, respectively). This idea is based on the well-known phenomenon of the superconvergence (see [8, 12, 28] for detail and more references) and has been used in many works by now, which include, for example, [11, 13, 14, 25, 26, 27, 29, 30], where quite effective error indicators, based on the superconvergence, have been proposed for both types of the error control and for various problems of elliptic type.

For our problem (1)–(3), it is natural to define such indicators as follows

$$\mathrm{I}_{gl}(u_h, G_h) := \int_\Omega \mathrm{A}(G_h(\nabla u_h) - \nabla u_h) \cdot (G_h(\nabla u_h) - \nabla u_h)\, dx, \tag{44}$$

and

$$\begin{aligned} \mathrm{I}_{loc}(u_h, G_h; v_\tau, G_\tau) := {} & E_0(u_h, v_\tau) + \\ & + \int_\Omega \mathrm{A}(G_h(\nabla u_h) - \nabla u_h) \cdot (G_\tau(\nabla v_\tau) - \nabla v_\tau)\, dx, \end{aligned} \tag{45}$$

for the global and local error control purposes, respectively. In the above, G_h and G_τ are some gradient averaging operators, see, e.g., works [8, 12] and references there for various definitions of them.

However, it is also well-known that the phenomenon of superconvergence is presented most strongly only in the case when the problem data is sufficiently smooth, which, in fact, considerably limits the quality and usage of the error indicators. Thus, in Section 5.3 we show that the indicators of type (44) and (45) completely fail, for example, if there are considerable jumps in the coefficients of the problem which is quite typical for most of real-life problems.

5.3 Numerical tests

In this section we present several numerical tests demonstrating the effectivity of the two-sided estimation procedures proposed in previous part of the paper, and also discuss the performance and failure of the error indicators (44), (45). All four tests are performed in planar domains, i.e., $d = 2$. For simplicity and purposes of comparison of the two-sided estimation and the error indicators we take the coefficient $c \equiv 0$, define the right-hand side function $f \equiv 10$, and consider the case of homogeneous Dirichlet boundary condition only, i.e., $\Gamma_D = \partial\Omega$ and $u_0 \equiv 0$ in all the tests. In this situation we need some upper bound only for one constant C_{Ω,Γ_D}, which can be easily done using Mikhlin's trick [19, p. 18]. The approximations $\bar{u}$ and $\bar{v}$ are assumed to be computed by the linear FEM, i.e., they are continuous piecewise linear functions defined by nodal values at vertices of computational meshes $\mathcal{T}_h$ (for main problem) and $\mathcal{T}_\tau$ (for adjoint problem) and denoted by u_h and v_τ, respectively, in what follows. The corresponding averaging operators are defined as follows (cf. [12]): the operator G_h maps the gradient $\nabla u_h = \left[\frac{\partial u_h}{\partial x_1}, \ldots, \frac{\partial u_h}{\partial x_d}\right]^T$, which is constant over each element of $\mathcal{T}_h$, into a vector-valued continuous piecewise affine function

$$G_h(\nabla u_h) = [G_h^1(\nabla u_h), \ldots, G_h^d(\nabla u_h)]^T,$$

by setting each its nodal value as the weighted mean (with respect to areas of elements) value of ∇u_h on all elements of the patch associated with corresponding node in the mesh $\mathcal{T}_h$. The operator G_τ is defined similarly.

For the sake of completeness we have computed the "exact errors" in all the tests using the so-called *reference solution* which is obtained by solving problem (1)–(3) on a very fine mesh with respect to the mesh $\mathcal{T}_h$.

5.3.1 Error control in global energy norm

Test 1: We consider problem (1)–(3) posed in a complicated planar domain Ω with a reentrant corner (see Fig. 1 (left)). Let A be equal to the unit matrix (denoted by the symbol I later on). The approximation $\bar{u} \equiv u_h$ is computed by FEM on the mesh $\mathcal{T}_h$ having 92 nodes. To obtain sufficiently accurate upper and lower bounds for the error in the corresponding energy norm, we employ several successive meshes. The behaviour of the two-sided estimates is presented in Fig. 1 (right) with the corresponding values given in Table 1: the upper bound is decreasing from 4.64 to 2.44, the lower bound grows from 1.22 to 1.74. We can clearly observe that the estimates are approaching each other in the process of error estimation, so the analyst can easily decide when to terminate the computational process. The error $\|\nabla(u-u_h)\|_\Omega^2 \approx 1.7514$. The gradient averaging indicator $\mathrm{I}_{gl} = 1.7547$, which is quite close to the exact error. Such a high effectivity of the indicator was really expected in this test due to the smoothness of the problem data.

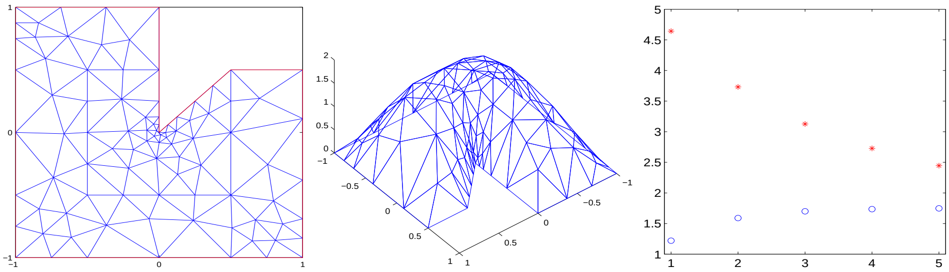

Figure 1: Solution domain Ω with computational mesh $\mathcal{T}_h$ having 92 nodes (left), the finite element approximation u_h (center), and optimization of the upper ("stars") and lower ("circles") estimates (right) in Test 1.

$M^\oplus$	$\|\nabla(u-u_h)\|_\Omega^2$	$M^\ominus$
4.6426	1.7514	1.2191
3.7313	1.7514	1.5893
3.1266	1.7514	1.6998
2.7266	1.7514	1.7347
2.4443	1.7514	1.7469

Table 1: Two-sided estimates versus the error $\|\nabla(u-u_h)\|_\Omega^2$ for Test 1.

Test 2: In this test we show that the global error indicator I_{gl} defined in (44), completely fails if the problem data is not sufficiently smooth. For this purpose we consider problem (1)–(3) posed in a simple square domain $\Omega := (-1,1) \times (-1,1)$ (see Fig. 2). Let the coefficient matrix A be defined as sketched in Fig. 2 below, i.e., A has high jumps in the entries (problem coefficients) along the diagonals and in the center of the quadratic domain Ω. The approximation $\bar{u} \equiv u_h$ is computed by FEM on the mesh $\mathcal{T}_h$ having 78 nodes (see Fig. 3 (left and center)).

For the indicator value we have $\mathrm{I}_{gl} = 17.9628$, which is quite different from the error $\|\nabla(u-u_h)\|_\Omega^2 \approx 0.8584$. However, our two-sided estimates provide with reliable estimation of the global error from above and below even in this case (cf. Fig. 3 (right) and Table 2).

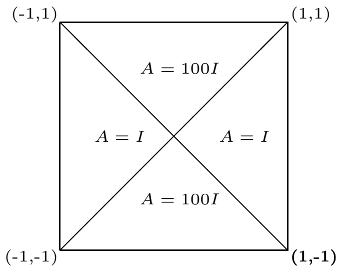

Figure 2: Definition of the coefficient matrix A for both, Test 2 and Test 4.

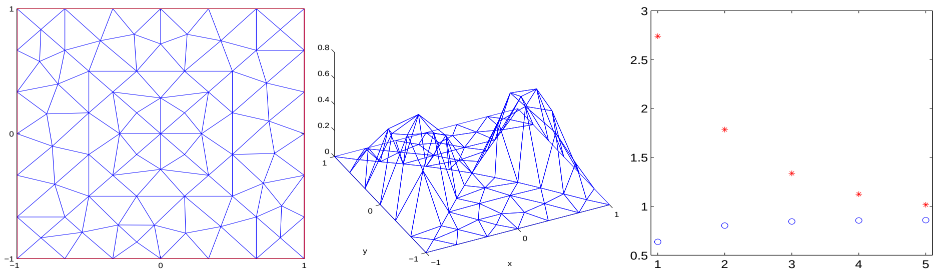

Figure 3: Solution domain Ω with computational mesh $\mathcal{T}_h$ having 78 nodes (left), the finite element approximation u_h (center), and optimization of the upper ("stars") and lower ("circles") bounds (right) in Test 2.

5.3.2 Error control in terms of linear functionals

Test 3: We consider the problem from Test 1 with the same approximation u_h. However, in this test, we want to control the deviation $u - u_h$ in terms of linear bounded functional ℓ, which we define as follows

$$\ell(w) = \int_\Omega \varphi w \, dx, \tag{46}$$

where $\varphi \in L_2(\Omega)$ and $\operatorname{supp} \varphi := \omega \subset \Omega$. The subdomain ω is taken in the neigbourhoud of the reentrant vertex (which always presents a zone of special interst for elliptic type boundary value problems), and is marked by the bold line in Fig. 4 (left), the weight-function in (46), $\varphi = 1$ in ω and vanishes

$M^\oplus$	$\|\nabla(u-u_h)\|_\Omega^2$	$M^\ominus$
2.7394	0.8584	0.6364
1.7842	0.8584	0.8025
1.3367	0.8584	0.8445
1.1241	0.8584	0.8551
1.0157	0.8584	0.8577

Table 2: Two-sided estimates versus the error $\|\nabla(u-u_h)\|_\Omega^2$ for Test 2.

outside of ω. Obviously, such defined φ belongs to the space $L_2(\Omega)$, i.e., the corresponding adjoint problem is uniquely solvable.

The optimization of two-sided estimates is presented in Fig. 4 (right) and Table 3. Different choices of computational meshes for the adjoint problem (with 20, 47, 66, 103, 151, and 208 nodes versus 92 nodes in $\mathcal{T}_h$) are used. The subindices 1, 2, 3 for $M^{\oplus}$ and $M^{\ominus}$ in Table 3 mean successive steps in the bounds' optimization. The error $\ell(u - u_h) \approx 0.0527$.

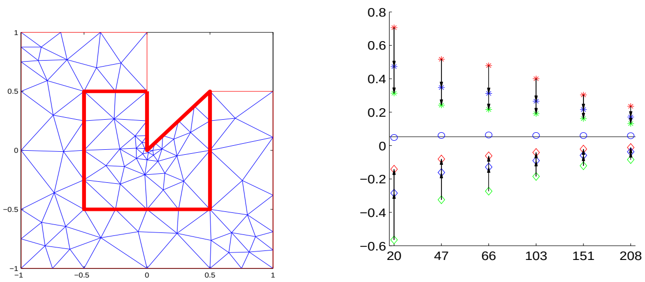

Figure 4: Solution domain Ω with ω marked by bold line, and mesh $\mathcal{T}_h$ (92 nodes) (left figure). The behaviour of the upper ("stars") and lower ("diamonds") estimates versus the error $\ell(u - u_h)$ ("line") and the indicator I_{loc} ("circles") values (right figure) for various choices of computational meshes for the corresponding adjoint problem in Test 3.

$\mathcal{T}_\tau$	$M_1^{\oplus}$	$M_2^{\oplus}$	$M_3^{\oplus}$	$\ell(u-u_h)$	$M_3^{\ominus}$	$M_2^{\ominus}$	$M_1^{\ominus}$	I_{loc}
20	0.7059	0.4730	0.3129	0.0527	-0.1419	-0.2850	-0.5648	0.0480
47	0.5165	0.3474	0.2416	0.0527	-0.0808	-0.1617	-0.3247	0.0604
66	0.4786	0.3121	0.2161	0.0527	-0.0619	-0.1286	-0.2734	0.0635
103	0.4003	0.2653	0.1907	0.0527	-0.0418	-0.0904	-0.1852	0.0603
151	0.3031	0.2153	0.1612	0.0527	-0.0213	-0.0585	-0.1209	0.0599
208	0.2352	0.1704	0.1313	0.0527	-0.0119	-0.0378	-0.0843	0.0588

Table 3: Two-sided estimates and the indicator I_{loc} versus the error $\ell(u-u_h)$ for Test 3.

In computation of two-sided bounds for the local error we clearly observe the following phenomenon (which also appears in Test 4): the upper and lower estimates converge to each other faster if a more fine computational mesh is used to approximately solve the adjoint problem. It can be explained by the fact that in the error decomposition (31) the second term E_1 is smaller with respect to the first term E_0 if the approximation $\bar{v}$ is more accurate (i.e., closer to the exact solution v). In this case the influence of the inaccuracy in two-sided estimation of terms like E_1 for all three problems (main, adjoint and mixed) is less crucial. However, computations of approximations (and the estimates' values) on very dense meshes in the adjoint problems maybe expensive, so certain balance between computational costs coming from the computations of the approximate solution in the adjoint problem and costs appearing in the process of optimization of the two-sided bounds proposed should be found, it may depend much on the software used for concrete calculations.

For completeness, we present in Fig. 5 examples of meshes used for computations of v_τ in the adjoint problem.

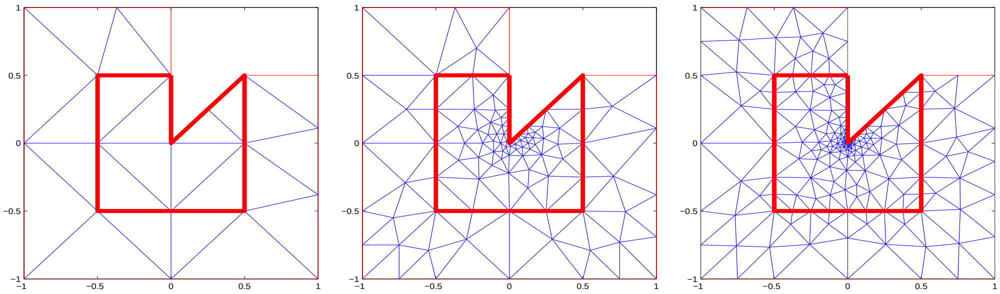

Figure 5: Examples of meshes for the adjoint problem with 20, 103, and 208 nodes used in Test 3.

Test 4: To demonstrate that the indicator (45), designed for the local error control, fails, we take the same problem as in Test 2. The zone of interest ω is the square $(-0.5, 0.5) \times (-0.5, 0.5)$, placed in the center of Ω (see Fig. 6 (left)).

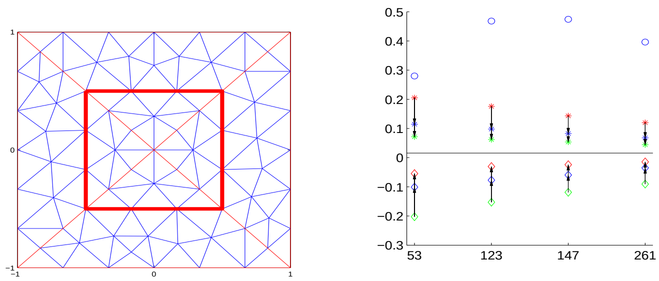

Figure 6: Solution domain Ω with ω marked by bold line and mesh $\mathcal{T}_h$ (78 nodes) (left figure). The behaviour of the upper ("stars") and lower ("diamonds") estimates versus the error $\ell(u - u_h)$ ("line") and the indicator I_{loc} ("circles") values (right figure) for various choices of computational meshes for the corresponding adjoint problem in Test 4.

Similarly to Test 3, we performed error estimation using several diferent meshes for the adjoint problem. The results are reported in Table 4, see also Fig. 6 (right). We clearly see that the values of I_{loc} are essentially bigger than $\ell(u - u_h) = 0.0157$ for all choices of the mesh for the adjoint problem. However, two-sided estimation procedures provide with reliable estimation of the local error $\ell(u - u_h)$ in all the cases. In Fig. 7 we present several meshes used in computation of approximations for the coresponding adjoint problem.

$\mathcal{T}_\tau$	$M_1^{\oplus}$	$M_2^{\oplus}$	$M_3^{\oplus}$	$\ell(u-u_h)$	$M_3^{\ominus}$	$M_2^{\ominus}$	$M_1^{\ominus}$	I_{loc}
53	0.2050	0.1148	0.0715	0.0157	-0.0547	-0.1008	-0.2026	0.2800
123	0.1757	0.0979	0.0620	0.0157	-0.0304	-0.0770	-0.1528	0.4681
147	0.1432	0.0823	0.0537	0.0157	-0.0240	-0.0594	-0.1187	0.4742
261	0.1197	0.0677	0.0441	0.0157	-0.0145	-0.0350	-0.0905	0.3961

Table 4: Two-sided bounds and the indicator I_{loc} versus the error $\ell(u-u_h)$ for Test 4.

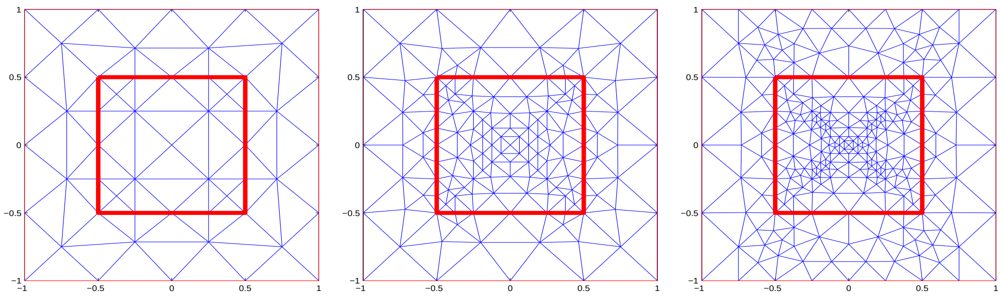

Figure 7: Examples of meshes for the adjoint problems with 53, 147, and 261 nodes used in Test 4.

6 Conclusions

From the construction of two-sided estimates and also from the tests presented, we see that both estimates can really be made arbitrarily close to the true errors. Such closeness only depends on the available resources (memory, computational velocity) of the concrete computer used for calculations. On the contrary, popular error indicators, based on the superconvergence effect, are not very relibale, especially if the problem data is not smooth enough. Nevertheless, due to simplicity of computation the indicators can be used, for example, for the first "rough" estimation of the error, or for the mesh adaptation purposes at initial steps, when the exact value of the error is not required.

The above presented ideas can be straightforwardly adapted to treating the other boundary conditions and the other linear elliptic problems (e.g., in linear elasticity).

The technologies of two-sided estimation proposed in this work can be easily coded and added as some block-checker to most of existing educational and industrial software products like MATLAB, FEMLAB, ANSYS, etc.

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