Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2005 A490

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Anders Björn, Niko Marola: *Moser iteration for (quasi)minimizers on metric spaces*; Helsinki University of Technology, Institute of Mathematics, Research Reports A490 (2005).

Abstract: *We study regularity properties of quasiminimizers of the p-Dirichlet integral on metric measure spaces. Our main objective is to adapt the Moser iteration technique to this setting. However, we have been able to run the Moser iteration fully only for minimizers. We prove Caccioppoli inequalities and local boundedness properties for quasisub- and quasisuperminimizers. This is done in metric spaces equipped with a doubling measure and supporting a weak $(1, p)$-Poincaré inequality without assuming completness of the metric space. New here seems to be that we do not assume completeness and only require a weak $(1, p)$-Poincaré inequality, rather than a weak $(1, q)$-Poincaré inequality for some $q < p$.*

We also provide an example which shows that the dilation constant from the weak Poincaré inequality is essential in the condition on the balls in the Harnack inequality. This fact seems to be overlooked in the earlier literature on nonlinear potential theory on metric spaces.

AMS subject classifications: Primary: 49N60; Secondary: 35J60, 49J27.

Keywords: Caccioppoli inequality, doubling measure, Harnack inequality, metric space, minimizer, Newtonian space, p-harmonic, Poincaré inequality, quasiminimizer, quasisubminimizer, quasisuperminimizer, Sobolev space, subminimizer, superminimizer.

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ISBN 951-22-7878-2
ISSN 0784-3143

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1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $1 < p < \infty$. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-quasiminimizer*, $Q \geq 1$, of the p-Dirichlet integral in Ω if for every open set $\Omega' \Subset \Omega$ and for all $\varphi \in W^{1,p}_0(\Omega')$ we have

$$\int_{\Omega'} |\nabla u|^p \, dx \leq Q \int_{\Omega'} |\nabla (u + \varphi)|^p \, dx.$$

In the Euclidean case the problem of minimizing the p-Dirichlet integral

$$\int_{\Omega} |\nabla u|^p \, dx$$

among all functions with given boundary values is equivalent to solving the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

A minimizer, or 1-quasiminimizer, is a weak solution of the p-Laplace equation. Being a weak solution is clearly a local property, however, being a quasiminimizer is not a local property. Hence, the theory for quasiminimizers usually differs from the theory for minimizers. Quasiminimizers were apparently first studied by Giaquinta–Giusti, see [13] and [14]. Quasiminimizers have been used as tools in studying regularity of minimizers of variational integrals. Namely, quasiminimizers have a rigidity that minimizers lack: the quasiminimizing condition applies to the whole class of variational integrals at the same time. For example, if a variational kernel $f(x, \nabla u)$ satisfies the inequalities

$$\alpha |h|^p \leq f(x, h) \leq \beta |h|^p$$

for some $0 < \alpha \leq \beta < \infty$, then the minimizers of $\int f(x, \nabla u)$ are quasiminimizers of the p-Dirichlet integral. Apart from this quasiminimizers have a fascinating theory in themselves. For more on quasiminimizers and their importance see the introduction in Kinnunen–Martio [27].

Giaquinta and Giusti [13], [14] proved several fundamental properties for quasiminimizers, including the interior regularity result that a quasiminimizer can be modified on a set of measure zero so that it becomes Hölder continuous. These results were extended to complete metric spaces by Kinnunen–Shanmugalingam [29].

In $\mathbf{R}^n$ minimizers of the p-Dirichlet integral are known to be locally Hölder continuous. This can be seen using either of the celebrated methods by De Giorgi (see [10]) and Moser (see [34] and [35]). Moser's method gives Harnack's inequality first and then Hölder continuity follows from this in a standard way, whereas De Giorgi first proves Hölder continuity from which Harnack's inequality follows. At the first sight it seems that Moser's technique is strongly based on the differential equation, whereas De Giorgi's method relies only on the minimization property. In Kinnunen–Shanmugalingam [29] De Giorgi's method was adapted to the metric setting. They proved that quasiminimizers are locally Hölder continuous, satisfy the strong maximum principle and Harnack's inequality. The space was assumed to be doubling in measure and to support a weak $(1, q)$-Poincaré inequality for some q with $1 < q < p$.

The purpose of this paper is twofold. First, we shall adapt Moser's iteration technique to the metric setting, and in particular show that the differential equation is not needed in the background for the Moser iteration. On the other hand, we will study quasiminimizers and show that certain estimates, which are interesting as such, extend to quasiminimizers as well. We have not been able to run the Moser iteration for quasiminimizers completely. Namely, there is one delicate step in the proof of Harnack's inequality using Moser's method. This is the so-called jumping over zero in the exponents related to the weak Harnack inequality. This is usually settled using the John–Nirenberg lemma for functions of bounded mean oscillation. More precisely, one have to show that a logarithm of a nonnegative quasisuperminimizer is a function of bounded mean oscillation. To prove this, the logarithmic Caccioppoli inequality is needed, which has been obtained only for minimizers. However, for minimizers we prove Harnack's inequality using the Moser iteration.

We will impose weaker requirements on the space than in Kinnunen–Shanmugalingam [29]. They assume that the space is complete, equipped with a doubling measure and supporting a weak $(1, q)$-Poincaré inequality for some $q < p$. We do not assume completeness and also only assume that the space supports a weak $(1, p)$-Poincaré inequality (doubling is still assumed). However, by a result of Keith and Zhong [22] a complete metric space equipped with a doubling measure that supports a weak $(1, p)$-Poincaré inequality, admits a weak $(1, q)$-Poincaré inequality. For examples of metric spaces equipped with a doubling measure supporting a Poincaré inequality, see, e.g., A. Björn [3].

As for completeness, in Kinnunen–Shanmugalingam [29] as well as in J. Björn [7], this was not assumed explicitly. However all three authors have informed us that both papers are written under the implicit assumption that the underlying metric space is complete. Thus this is the first paper, as far as we know, in which regularity results for harmonic and p-harmonic functions are obtained in a noncomplete setting. In fact this also applies to the linear case. Linear potential theory has been developed axiomatically in several different ways, but all such theories, as far as we know, assume the underlying space to be locally compact.

At the end of the paper we provide an example which shows that the dilation constant from the weak Poincaré inequality is essential in the condition on the balls in the weak Harnack inequality (Theorem 9.2) and Harnack's inequality (Theorem 9.3). This fact is overlooked in certain results and proofs of [29]. In addition, certain quantitative statements in Kinnunen–Martio [26], [27] and A. Björn [3] need to be modified according to our example.

The paper is organized as follows. In the second section we impose requirements for the measure and the third section focuses on the notation, definitions and concepts used throughout this paper. The fourth section explores the relationship between alternative definitions of Newtonian spaces with zero boundary values in the setting of noncomplete metric spaces. Section 5 introduces Sobolev–Poincaré inequalities crucial for us in what follows and in Section 6 we will prove the equivalence of different definitions for quasi(super)minimizers. The next two sections are devoted to Caccioppoli inequalities and weak Harnack inequalities. In particular, local boundedness results for quasisub- and quasisuperminimizers are proved. In Section 9 only minimizers, i.e. 1-quasiminimizers, are studied. We prove Harnack's inequality for minimizers and as a corollary Liouville's theorem. In the final section we give a counterexample motivating the results in Section 9.

Acknowledgements. The authors are grateful to Jana Björn, Juha Kinnunen and Nageswari Shanmugalingam for their interest and encouragement. The first author is supported by the Swedish Research Council, while the second author acknowledges the support of the Finnish Academy of Science and Letters, Vilho, Yrjö and Kalle Väisälä Foundation. These results were partially obtained while the second author was visiting Linköpings universitet in February 2005.

2. Doubling

We assume throughout the paper that $1 < p < \infty$ and that $X = (X, d, \mu)$ is a metric space endowed with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$ (we make the convention that balls are nonempty and open). Let us here also point out that at the end of Section 3 we further assume that X supports a weak $(1, p)$-Poincaré inequality and that μ is doubling, which is then assumed throughout the rest of the paper.

We emphasize that the σ-algebra on which μ is defined is obtained by the completion of the Borel σ-algebra. We further extend μ as an outer measure on X, so that for an arbitrary set $A \subset X$ we have

$$\mu(A) = \inf\{\mu(E) : E \supset A \text{ is a Borel set}\}.$$

It is more or less immediate that μ is a Borel regular measure, in the sense defined by Federer [11], Section 2.2.3, i.e. for every $E \subset X$ there is a Borel set $B \supset E$ such that $\mu(E) = \mu(B)$. If $E \subset X$ is measurable, then there exist Borel sets A and B such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. (Note that Rudin [36] has a more restrictive definition of Borel regularity which is not always fulfilled for our spaces.)

The measure μ is said to be *doubling* if there exists a constant $C_\mu \geq 1$, called the *doubling constant* of μ, such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X,

$$\mu(2B) \leq C_\mu \mu(B),$$

where $\lambda B = B(x_0, \lambda r)$. By the doubling property, if $B(y,R)$ is a ball in X, $z \in B(y,R)$ and $0 < r \le R < \infty$, then

$$\frac{\mu(B(z,r))}{\mu(B(y,R))} \ge C\Big(\frac{r}{R}\Big)^s \tag{2.1}$$

for $s = \log_2 C_\mu$ and some constant C only depending on C_μ. The exponent s serves as a counterpart of dimension related to the measure.

A metric space is *doubling* if there exists a constant $C < \infty$ such that every ball $B(z,r)$ can be covered by at most C balls with radii $\frac{1}{2}r$. Alternatively and equivalently, for every $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that every ball $B(z,r)$ can be covered by at most $C(\varepsilon)$ balls with radii εr. It is now easy to see that every bounded set in a doubling metric space is totally bounded.

A metric space equipped with a doubling measure is doubling, and conversely any complete doubling metric space can be equipped with a doubling measure. See Heinonen [20], Section 10.13, for more on doubling metric spaces.

The following proposition is well known. However, since it does not seem to appear explicitly in the literature, we give a short proof here.

Proposition 2.1. *Let Y be a doubling metric space. Then Y is proper* (*i.e., closed and bounded subsets of Y are compact*) *if and only if Y is complete.*

Proof. Assume that Y is proper and take a Cauchy sequence $\{x_i\}_{i=1}^\infty$. Then for a sufficiently large radius $r > 0$, $x_i \in \overline{B}(x_1, r) \subset Y$. By the properness of Y this set is compact and the sequence has a limit in Y.

Conversely, let Y be complete and M be a closed and bounded subset of Y. Then M is totally bounded, and hence compact, see, e.g., Rudin [37], Theorem A4. □

3. Newtonian spaces

In this paper a *path* in X is a rectifiable nonconstant continuous mapping from a compact interval. (For us only such paths will be interesting, in general a path is a continuous mapping from an interval.) A path can thus be parameterized by arc length ds.

Definition 3.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all paths $\gamma : [0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds \tag{3.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (3.1) holds for p-almost every path, then g is a *p-weak upper gradient* of f.

By saying that (3.1) holds for p-almost every path we mean that it fails only for a path family with zero p-modulus, see Definition 2.1 in Shanmugalingam [38]. It is implicitly assumed that $\int_\gamma g\,ds$ is defined (with a value in $[0,\infty]$) for p-almost every path.

If $g \in L^p(X)$ is a p-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \to g$ in $L^p(X)$, see Lemma 2.4 in Koskela–MacManus [31].

If f has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \le g$ μ-a.e., see Corollary 3.7 in Shanmugalingam [39]. The minimal p-weak upper gradient can be given by the formula

$$g_f(x) := \inf_g \limsup_{r\to 0+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g\,d\mu,$$

where the infimum is taken over all upper gradients $g \in L^p(X)$ of f, see Lemma 2.3 in J. Björn [7].

Lemma 3.2. *Let u and v be functions with upper gradients in $L^p(X)$. Then $g_u\chi_{\{u>v\}} + g_v\chi_{\{v\ge u\}}$ is a minimal p-weak upper gradient of $\max\{u,v\}$, and $g_v\chi_{\{u>v\}} + g_u\chi_{\{v\ge u\}}$ is a minimal p-weak upper gradient of $\min\{u,v\}$.*

This lemma was proved in Björn–Björn [5], Lemma 3.2, and a different proof was given in Marola [32], Lemma 3.5.

Following Shanmugalingam [38], we define a version of Sobolev spaces on the metric space X.

Definition 3.3. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p\,d\mu + \inf_g \int_X g^p\,d\mu\right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u-v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Shanmugalingam [38].

Definition 3.4. The *capacity* of a set $E \subset X$ is the number

$$C_p(E) = \inf \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

The capacity is countably subadditive. For this and other properties as well as equivalent definitions of the capacity we refer to Kilpeläinen–Kinnunen–Martio [23] and Kinnunen–Martio [24], [25].

We say that a property regarding points in X holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [38] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$.

Definition 3.5. We say that X supports a *weak* $(1,p)$*-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all measurable functions f on X and for all upper gradients g of f,

$$\fint_B |f - f_B| \, d\mu \leq C(\operatorname{diam} B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}, \tag{3.2}$$

where $f_B := \fint_B f \, d\mu := \int_B f \, d\mu / \mu(B)$.

By the Hölder inequality it is easy to see that if X supports a weak $(1,p)$-Poincaré inequality, then it supports a weak $(1,q)$-Poincaré inequality for every $q > p$. In the above definition of Poincaré inequality we can equivalently assume that g is a p-weak upper gradient—see the comments above.

Let us throughout the rest of the paper assume that X supports a weak $(1,p)$-Poincaré inequality and that μ is doubling.

It then follows that Lipschitz functions are dense in $N^{1,p}(X)$ and that the functions in $N^{1,p}(X)$ are quasicontinuous, see [38]. This means that in the Euclidean setting, $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space as defined on p. 96 of Heinonen–Kilpeläinen–Martio [21].

We end this section by recalling that $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$.

Unless otherwise stated, the letter C denotes various positive constants whose exact values are unimportant and may vary with each usage.

4. Newtonian spaces with zero boundary values

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. We let for a measurable set $E \subset X$,

$$N_0^{1,p}(E) = \{f|_E : f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus E\}.$$

One can replace the assumption "$f = 0$ on $X \setminus E$" with "$f = 0$ q.e. on $X \setminus E$" without changing the obtained space $N_0^{1,p}(E)$. Note that if $C_p(X \setminus E) = 0$, then $N_0^{1,p}(E) = N^{1,p}(E)$. The space $N_0^{1,p}(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space, see Theorem 4.4 in Shanmugalingam [39].

The space $N_0^{1,p}(E)$ is however not the only natural candidate for a Newtonian space with zero boundary values, another natural candidate is $N_b^{1,p}(\Omega)$, where we from now on assume that $\Omega \subset X$ is open.

Definition 4.1. We write $E \dot{\Subset} \Omega$ if E is bounded and $\operatorname{dist}(E, X \setminus \Omega) > 0$. We also let $\operatorname{Lip}_b(\Omega) = \{f \in \operatorname{Lip}(X) : \operatorname{supp} f \dot{\Subset} \Omega\}$, and $N_b^{1,p}(\Omega) = \overline{\operatorname{Lip}_b(\Omega)}$.

The closures here and below are with respect to the $N^{1,p}$-norm, and it is immediate that $N_b^{1,p}(\Omega)$ is a Banach space. (By the way, the letter "b" has been chosen by its proximity to "c" and because of the word "bounded".)

Note that if X is complete, then $E \dot{\Subset} \Omega$ if and only if $E \Subset \Omega$, and $\operatorname{Lip}_c(\Omega) = \operatorname{Lip}_b(\Omega)$. (Recall that $E \Subset \Omega$ if $\overline{E}$ is a compact subset of Ω, and that $\operatorname{Lip}_c(\Omega) = \{f \in \operatorname{Lip}(X) : \operatorname{supp} f \Subset \Omega\}$.) When X is complete, we know that $N_b^{1,p}(\Omega) = N_0^{1,p}(\Omega)$, see Shanmugalingam [39], Theorem 4.8.

The equality $N_0^{1,p}(\Omega) = N_b^{1,p}(\Omega)$ goes under the name "spectral synthesis" in the literature. The history goes back to Beurling and Deny; Hedberg [18] showed the corresponding result for higher order Sobolev spaces on $\mathbf{R}^n$ (modulo the Kellogg property which at that time was only known to hold for $p > 2 - 1/n$, but was later proved in general by Wolff); see Adams–Hedberg [1], Section 9.13, for a historical account as well as an explanation of the name spectral synthesis. For spectral synthesis in very general function spaces on $\mathbf{R}^n$, including, e.g., Besov and Lizorkin–Triebel spaces, see Hedberg–Netrusov [19].

In the noncomplete case we have been unable to prove spectral synthesis. Let us explain the difficulty: In the proof of Theorem 4.8 in Shanmugalingam [39] she first proves Lemma 4.10, and this later proof carries over verbatim to the noncomplete case. However, if $u \in N_0^{1,p}(\Omega)$, we do not see how one can conclude that $\operatorname{supp} \varphi_k \dot{\Subset} \Omega$, where φ_k is given in the statement of Lemma 4.10. This fact is the main purpose of Lemma 4.10 and it is used in the subsequent proof of Theorem 4.8.

The following result is true.

Proposition 4.2. *It is true that*

$$\begin{aligned} N_b^{1,p}(\Omega) &= \overline{\operatorname{Lip}_0(\Omega)} = \overline{\{f \in N^{1,p}(X) : \operatorname{supp} f \dot{\Subset} \Omega\}} \\ &= \overline{\{f \in N^{1,p}(X) : \operatorname{dist}(\operatorname{supp} f, X \setminus \Omega) > 0\}}. \end{aligned}$$

Here, $\operatorname{Lip}_0(\Omega) := N_0^{1,p}(\Omega) \cap \operatorname{Lip}(X)$. If Ω is bounded, then $\operatorname{Lip}_0(\Omega) = \{f \in \operatorname{Lip}(X) : f = 0 \text{ outside of } \Omega\}$.

To prove this proposition we need a lemma which will also be useful to us later.

Lemma 4.3. *Let $u \in N_0^{1,p}(\Omega)$ have bounded support and let $\varepsilon > 0$. Then there is a function $\psi \in \operatorname{Lip}_b(X)$ and a set E such that $E \subset \{x : \operatorname{dist}(x, \Omega) < \varepsilon\}$, $\mu(E) < \varepsilon$, $\psi = u$ in $X \setminus E$ and $\|\psi - u\|_{N^{1,p}(X)} < \varepsilon$.*

Note in particular that $\psi = 0$ in $X \setminus (\Omega \cup E)$. (We consider a function in $N_0^{1,p}(\Omega)$ to be identically 0 outside of Ω.)

Proof. Assume first that $0 \le u \le 1$.

Let A be the set of non-Lebesgue points of u, which has measure 0, see, e.g., Heinonen [20], Theorem 1.8. Since $u = 0$ outside of Ω we get immediately that $A \subset \overline{\Omega}$.

Let $\tau > 0$. In the construction given in the proof of Theorem 2.12 in Shanmugalingam [39], one find a set E_τ such that

$$\tau^p \mu(E_\tau) \to 0, \quad \text{as } \tau \to \infty,$$

and a $C\tau$-Lipschitz function u_τ on $X \setminus E_\tau$. It is observed that $u_\tau = u$ on $X \setminus (E_\tau \cup A)$.

In the proof in [39] one then extends u_τ as a $C\tau$-Lipschitz function on X. There are several ways to do this, but we here prefer to choose this extension to be the minimal nonnegative $C\tau$-Lipschitz extension to X. It is thus given by (we abuse notation and call also the extension u_τ)

$$u_\tau(x) := \max\{u_\tau(y) - C\tau d(x,y) : y \in X \setminus E_\tau\}_+, \quad x \in X.$$

It follows that $u_\tau(x) = 0$ when $\operatorname{dist}(x,\Omega) \ge 1/C\tau$.

Choose now τ so large that $\mu(E_\tau) < \varepsilon$, $1/C\tau < \varepsilon$, and $\|u_\tau - u\|_{N^{1,p}(X)} < \varepsilon$. Letting $\psi = u_\tau$ and

$$E = \{x \in E_\tau \cup A : \operatorname{dist}(x,\Omega) < \varepsilon\},$$

gives the desired conclusion in the case when $0 \le u \le 1$, and hence also in the case when u is nonnegative and bounded.

Let next u be arbitrary. By Lemma 4.9 in [39] we can find $k > 0$ such that $\mu(\{x : |u(x)| > k\}) < \varepsilon$ and $\|u - u_k\|_{N^{1,p}(X)} < \varepsilon$, where $u_k = \max\{\min\{u,k\},-k\}$. Applying this lemma to $(u_k)_\pm$ we find functions $\psi_\pm \in \operatorname{Lip}_b(\Omega)$ and sets $E_\pm \subset \{x : \operatorname{dist}(x,\Omega) < \varepsilon\}$ such that $\mu(E_\pm) < \varepsilon$, $\psi_\pm = (u_k)_\pm$ on $X \setminus E_\pm$ and $\|\psi_\pm - (u_k)_\pm\|_{N^{1,p}(X)} < \varepsilon$. Letting $\psi = \psi_+ - \psi_-$ and $E = E_+ \cup E_- \cup \{x : |u(x)| > k\}$ completes the proof. $\square$

Lemma 4.4. *Let $u \in N_0^{1,p}(\Omega)$ and $\varepsilon > 0$. Then there is a function $\psi \in \operatorname{Lip}_b(X)$ and a set E such that $E \subset \{x : \operatorname{dist}(x,\Omega) < \varepsilon\}$, $\mu(E) < \varepsilon$, $\psi = 0$ in $X \setminus (\Omega \cup E)$, $|\psi(x) - u(x)| < \varepsilon$ for $x \in X \setminus E$ and $\|\psi - u\|_{N^{1,p}(X)} < \varepsilon$.*

Proof. By Lemma 2.14 in Shanmugalingam [39] we can find $u' \in N_0^{1,p}(\Omega)$ with bounded support such that $\|u - u'\|_{N^{1,p}(X)} < \tfrac{1}{2}\varepsilon$ and $\mu(\{x : |u'(x) - u(x)| \ge \tfrac{1}{2}\varepsilon\}) < \tfrac{1}{2}\varepsilon$. Applying Lemma 4.3 to u' and $\tfrac{1}{2}\varepsilon$ gives a function ψ and a set E' such that $E' \subset \{x : \operatorname{dist}(x,\Omega) < \varepsilon\}$, $\mu(E') < \tfrac{1}{2}\varepsilon$, $\psi = u'$ on $X \setminus E'$ and $\|\psi - u'\|_{N^{1,p}(X)} < \tfrac{1}{2}\varepsilon$. Letting $E = E' \cup \{x : |u'(x) - u(x)| \ge \tfrac{1}{2}\varepsilon\}$ concludes the proof. $\square$

Proof of Proposition 4.2. The inclusions $N_b^{1,p}(\Omega) \subset \overline{\operatorname{Lip}_0(\Omega)}$ and

$$\begin{aligned} N_b^{1,p}(\Omega) &\subset \overline{\{f \in N^{1,p}(X) : \operatorname{supp} f \Subset \Omega\}} \\ &\subset \overline{\{f \in N^{1,p}(X) : \operatorname{dist}(\operatorname{supp} f, X \setminus \Omega) > 0\}} \end{aligned}$$

are clear.

Let $\varphi \in \mathrm{Lip}_0(\Omega)$ and $\varepsilon > 0$. By approximating as in Lemma 2.14 in Shanmugalingam [39] we find a function $\psi \in \mathrm{Lip}_0(\Omega)$ with bounded support and such that $\|\varphi - \psi\|_{N^{1,p}(X)} < \varepsilon$. Let

$$\psi_k = (\psi_+ - 1/k)_+ - (\psi_- - 1/k)_+.$$

Then $\|\psi - \psi_k\|_{N^{1,p}(X)} \to 0$, as $k \to \infty$, and $\psi_k \in \mathrm{Lip}_b(\Omega)$. Thus $\varphi \in N^{1,p}_b(\Omega)$.

Let finally $\varphi \in N^{1,p}(X)$ be such that $\operatorname{dist}(\operatorname{supp}\varphi, X \setminus \Omega) > 0$. Let $\varepsilon = \frac{1}{3}\operatorname{dist}(\operatorname{supp}\varphi, X \setminus \Omega)$ and $\Omega' = \{x : \operatorname{dist}(x, \operatorname{supp}\varphi) < \varepsilon\}$. By Lemma 4.4 we find a Lipschitz function ψ such that $\|\psi - \varphi\|_{N^{1,p}(X)} < \varepsilon$ and $\operatorname{supp}\psi \subset \{x : \operatorname{dist}(x, \operatorname{supp}\varphi) < 2\varepsilon\}$, i.e. $\psi \in \mathrm{Lip}_b(\Omega)$. Thus $\varphi \in N^{1,p}_b(\Omega)$. □

5. Sobolev–Poincaré inequalities

In this section we introduce certain Sobolev–Poincaré inequalities which will be crucial in what follows.

A result of Hajłasz–Koskela [16] (see also Hajłasz–Koskela [17]) shows that in a doubling measure space a weak $(1,p)$-Poincaré inequality implies a *Sobolev-Poincaré inequality.* More precisely, there exists a constant $C > 0$ only depending on p, C_μ and the constants in the weak Poincaré inequality, such that

$$\left(\fint_{B(z,r)} |f - f_{B(z,r)}|^{\kappa p}\, d\mu\right)^{1/\kappa p} \le Cr\left(\fint_{B(z,5\lambda r)} g_f^p\, d\mu\right)^{1/p}, \tag{5.1}$$

where $\kappa = s/(s-p)$ if $1 < p < s$ and $\kappa = 2$ if $p \ge s$, for all balls $B(z,r) \subset X$, for all integrable functions f on $B(z,r)$ and for minimal p-weak upper gradients g_f of f.

We will also need an inequality for Newtonian functions with zero boundary values. If $f \in N^{1,p}_0(B(z,r))$, then there exists a constant $C > 0$ only depending on p, C_μ and the constants in the weak Poincaré inequality, such that

$$\left(\fint_{B(z,r)} |f|^{\kappa p}\, d\mu\right)^{1/\kappa p} \le Cr\left(\fint_{B(z,r)} g_f^p\, d\mu\right)^{1/p} \tag{5.2}$$

for every ball $B(z,r)$ with $r \le \frac{1}{3}\operatorname{diam} X$. For this result we refer to Kinnunen–Shanmugalingam [29], equation (2.6). In [29] it was assumed that the space supports a weak $(1,q)$-Poincaré inequality for some q with $1 < q < p$. However, the assumption is not used in the proof of (5.2).

6. Quasi(super)minimizers

This section is devoted to quasiminimizers, and in particular to quasisuperminimizers. We prove the equivalence of different definitions for quasisuperminimizers.

Definition 6.1. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-quasiminimizer* in Ω if for all open $\Omega' \Subset \Omega$ and all $\varphi \in N^{1,p}_0(\Omega')$ we have

$$\int_{\Omega'} g_u^p \, d\mu \le Q \int_{\Omega'} g_{u+\varphi}^p \, d\mu. \tag{6.1}$$

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-quasisuperminimizer* in Ω if (6.1) holds for all nonnegative $\varphi \in N^{1,p}_0(\Omega')$, and a *$Q$-quasisubminimizer* in Ω if (6.1) holds for all nonpositive $\varphi \in N^{1,p}_0(\Omega')$.

We say that $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if $f \in N^{1,p}(\Omega')$ for every open $\Omega' \Subset \Omega$ (or equivalently that $f \in N^{1,p}(E)$ for every $E \Subset \Omega$). Observe that since X is not assumed to be proper, it is not enough to require that for every $x \in \Omega$ there is an $r > 0$ such that $f \in N^{1,p}(B(x,r))$.

A function is a Q-quasiminimizer in Ω if and only if it is both a Q-quasisubminimizer and a Q-quasisuperminimizer in Ω (this is most easily seen by writing $\varphi = \varphi_+ - \varphi_-$ and using (d) below).

When $Q = 1$, we drop "quasi" from the notation and say, e.g., that a minimizer is a 1-quasiminimizer.

Proposition 6.2. *Let $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Then the following are equivalent*:

(a) *The function u is a Q-quasisuperminimizer in Ω*;

(b) *For all open $\Omega' \Subset \Omega$ and all nonnegative $\varphi \in N^{1,p}_0(\Omega')$ we have*

$$\int_{\overline{\Omega}'} g_u^p \, d\mu \le Q \int_{\overline{\Omega}'} g_{u+\varphi}^p \, d\mu;$$

(c) *For all μ-measurable sets $E \Subset \Omega$ and all nonnegative $\varphi \in N^{1,p}_0(E)$ we have*

$$\int_{E} g_u^p \, d\mu \le Q \int_{E} g_{u+\varphi}^p \, d\mu;$$

(d) *For all nonnegative $\varphi \in N^{1,p}(\Omega)$ with $\operatorname{supp} \varphi \Subset \Omega$ we have*

$$\int_{\varphi \ne 0} g_u^p \, d\mu \le Q \int_{\varphi \ne 0} g_{u+\varphi}^p \, d\mu;$$

(e) *For all nonnegative $\varphi \in N^{1,p}(\Omega)$ with $\operatorname{supp} \varphi \Subset \Omega$ we have*

$$\int_{\operatorname{supp} \varphi} g_u^p \, d\mu \le Q \int_{\operatorname{supp} \varphi} g_{u+\varphi}^p \, d\mu;$$

(f) *For all nonnegative $\varphi \in \mathrm{Lip}_b(\Omega)$ we have*

$$\int_{\varphi \ne 0} g_u^p \, d\mu \le Q \int_{\varphi \ne 0} g_{u+\varphi}^p \, d\mu;$$

(g) *For all nonnegative $\varphi \in \mathrm{Lip}_b(\Omega)$ we have*

$$\int_{\operatorname{supp} \varphi} g_u^p \, d\mu \le Q \int_{\operatorname{supp} \varphi} g_{u+\varphi}^p \, d\mu.$$

Remark 6.3. (1) If we omit "super" from (a) and "nonnegative" from (b)–(g) we have a corresponding characterization for Q-quasiminimizers. The proof of these equivalences is the same as the proof below.

(2) In the case when X is complete, Kinnunen–Martio [27], Lemmas 3.2, 3.4 and 6.2, gave the characterizations (a)–(d), and all of the statements above as well as some more were shown to be equivalent in A. Björn [2].

(3) It is easy to see from the definition that if $\Omega_1 \subset \Omega_2 \subset ... \subset \Omega$ and for every $\Omega' \dot{\Subset} \Omega$ there is $\Omega_j \supset \Omega'$, then u is a Q-quasisuperminimizer in Ω if and only if u is a Q-quasisuperminimizer in Ω_j for every j. (Observe that when X is complete it is equivalent to just require that $\Omega_1 \subset \Omega_2 \subset ... \subset \Omega$ and $\Omega = \bigcup_{j=1}^\infty \Omega_j$, it then follows by compactness that if $\Omega' \Subset \Omega$ then there is $\Omega_j \supset \Omega'$.)

(4) On $\mathbf{R}^n$ it is known that a function u is a superminimizer in an open set Ω if and only if for every $x \in \Omega$ there is $r > 0$ such that u is a superminimizer in $B(x, r)$; this is sometimes called the sheaf property. To prove the nontrivial implication one uses the p-Laplace equation together with partition of unity; the same can be done for Cheeger superminimizers in complete doubling metric spaces supporting a Poincaré inequality (see J. Björn [8]).

For our superminimizers defined using upper gradients we do not have a corresponding differential equation (and cannot use a partition of unity argument). It is therefore unknown if the sheaf property holds for our superminimizers, even if we restrict ourselves to complete metric spaces.

Quasisuperminimizers do not form sheaves even in $\mathbf{R}^n$ (in fact not even on $\mathbf{R}$).

(5) Let us for the moment make the following definition. A function $u \in N^{1,p}_{\text{loc}}(\Omega)$ is a *strong quasisuperminimizer* in Ω if for all nonnegative $\varphi \in N^{1,p}_b(\Omega)$ (or $N^{1,p}_0(\Omega)$) we have

$$\int_{\varphi \neq 0} g_u^p \, d\mu \leq Q \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu.$$

When X is complete this is equivalent to our definition, see A. Björn [2]. (Moreover, this definition was used by Ziemer [41] in $\mathbf{R}^n$, but he was no doubt aware of the equivalence in this case.)

In noncomplete metric spaces we have been unable to show that quasisuperminimizers are strong quasisuperminimizers. For the purposes of this paper our weaker assumption is enough, but it could happen that for some other results about quasisuperminimizers (e.g. in the theory of boundary regularity) the right condition is to require the functions involved to be strong quasisuperminimizers.

For strong quasisuperminimizers the property described in (3) above does not hold, unless the definitions indeed are equivalent: Let u be a quasisuperminimizer in Ω which is not a strong quasisuperminimizer. Let further

$$\Omega_j = \{x : d(x, y) < j \text{ and } \operatorname{dist}(x, X \setminus \Omega) \geq 1/j\},$$

where $y \in X$ is some fixed point. Then u is a strong quasisuperminimizer in Ω_j for every j.

Proof. (a) $\Rightarrow$ (c) (This is similar to the proof of Lemma 3.2 in Kinnunen–Martio [27].) Let $\varepsilon > 0$. By the regularity of the measure we can find an open set Ω' such that $E \subset \Omega' \Subset \Omega$ and

$$\int_{\Omega' \setminus E} g_{u+\varphi}^p \, d\mu < \frac{\varepsilon}{Q}.$$

Since $\varphi \in N_0^{1,p}(\Omega')$ we have

$$\begin{aligned}\int_E g_u^p \, d\mu \le \int_{\Omega'} g_u^p \, d\mu &\le Q \int_{\Omega'} g_{u+\varphi}^p \, d\mu \\ &= Q \int_E g_{u+\varphi}^p \, d\mu + Q \int_{\Omega' \setminus E} g_{u+\varphi}^p \, d\mu \le Q \int_E g_{u+\varphi}^p \, d\mu + \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof of this implication.

(c) $\Rightarrow$ (d) $\Rightarrow$ (f) This is trivial.

(f) $\Rightarrow$ (a) Let $\Omega' \Subset \Omega$ be open and $\varphi \in N_0^{1,p}(\Omega')$ be nonnegative. Let $0 < \varepsilon < \frac{1}{2} \operatorname{dist}(\Omega', X \setminus \Omega)$. By Lemma 4.3, we can find a nonnegative Lipschitz function ψ and a set $E \subset \{x : d(x, \Omega') < \varepsilon\} \Subset \Omega$ such that $\mu(E) < \varepsilon$, $\psi = \varphi$ in $X \setminus E$ and $\|\psi - \varphi\|_{N^{1,p}(X)} < \varepsilon / Q^{1/p}$.

Let $A = \{x : \psi(x) \ne 0\}$. Since $g_u = g_{u+\psi}$ μ-a.e. outside of A and $\operatorname{supp} \psi \Subset \Omega$ we find that

$$\begin{aligned}\int_{\Omega'} g_u^p \, d\mu &\le \int_{\psi \ne 0} g_u^p \, d\mu + \int_{\Omega' \setminus A} g_u^p \, d\mu \\ &\le Q \int_{\psi \ne 0} g_{u+\psi}^p \, d\mu + \int_{\Omega' \setminus A} g_{u+\psi}^p \, d\mu \le Q \int_{\Omega' \cup A} g_{u+\psi}^p \, d\mu.\end{aligned}$$

Thus

$$\begin{aligned}\biggl(\int_{\Omega'} g_u^p \, d\mu \biggr)^{1/p} &\le \biggl(Q \int_{\Omega' \cup A} g_{u+\varphi}^p \, d\mu \biggr)^{1/p} + \biggl(Q \int_{\Omega' \cup A} g_{\psi-\varphi}^p \, d\mu \biggr)^{1/p} \\ &\le \biggl(Q \int_{\Omega'} g_{u+\varphi}^p \, d\mu + Q \int_{A \setminus \Omega'} g_{u+\varphi}^p \, d\mu \biggr)^{1/p} + \varepsilon.\end{aligned}$$

Since $\int_{A \setminus \Omega'} g_{u+\varphi}^p \, d\mu \to 0$, as $\varepsilon \to 0$, we obtain the required estimate.

(a) $\Rightarrow$ (b) Since $g_u = g_{u+\varphi}$ μ-a.e. on $\partial\Omega'$, we get

$$\begin{aligned}\int_{\overline{\Omega}'} g_u^p \, d\mu &= \int_{\partial\Omega'} g_u^p \, d\mu + \int_{\Omega'} g_u^p \, d\mu \\ &\le \int_{\partial\Omega'} g_{u+\varphi}^p \, d\mu + Q \int_{\Omega'} g_{u+\varphi}^p \, d\mu \le Q \int_{\overline{\Omega}'} g_{u+\varphi}^p \, d\mu.\end{aligned}$$

(b) $\Rightarrow$ (e) Let $\varepsilon > 0$. By the regularity of the measure we can find $\delta > 0$ such that $\Omega'' := \{x : \operatorname{dist}(x, \operatorname{supp} \varphi) < 2\delta\} \Subset \Omega$ and

$$\int_{\Omega'' \setminus \operatorname{supp} \varphi} g_{u+\varphi}^p \, d\mu < \frac{\varepsilon}{Q}.$$

Letting $\Omega' := \{x : \operatorname{dist}(x, \operatorname{supp}\varphi) < \delta\}$ we have

$$\begin{aligned}\int_{\operatorname{supp}\varphi} g_u^p \, d\mu &\le \int_{\overline{\Omega}'} g_u^p \, d\mu \\ &\le Q \int_{\overline{\Omega}'} g_{u+\varphi}^p \, d\mu \\ &= Q \int_{\operatorname{supp}\varphi} g_{u+\varphi}^p \, d\mu + Q \int_{\overline{\Omega}' \setminus \operatorname{supp}\varphi} g_{u+\varphi}^p \, d\mu \\ &\le Q \int_{\operatorname{supp}\varphi} g_{u+\varphi}^p \, d\mu + \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof of this implication.

(e) $\Rightarrow$ (g) This is trivial.

(g) $\Rightarrow$ (f) Let $\varphi_j = (\varphi - 1/j)_+$. We get

$$\begin{aligned}\int_{\varphi\neq 0} g_u^p \, d\mu &= \lim_{j\to\infty} \int_{\operatorname{supp}\varphi_j} g_u^p \, d\mu \le Q \lim_{j\to\infty} \int_{\operatorname{supp}\varphi_j} g_{u+\varphi_j}^p \, d\mu \\ &\le Q \lim_{j\to\infty} \int_{\varphi\neq 0} g_{u+\varphi_j}^p \, d\mu = Q \int_{\varphi\neq 0} g_{u+\varphi}^p \, d\mu. \qquad \square\end{aligned}$$

The following lemma is a crucial fact about quasisuperminimizers.

Lemma 6.4. *Let u_j be a Q_j-quasisuperminimizer, $j = 1, 2$. Then $\min\{u_1, u_2\}$ is a $\min\{Q_1 + Q_2, Q_1 Q_2\}$-quasisuperminimizer.*

This is proved in Kinnunen–Martio [27], Lemmas 3.6, 3.7 and Corollary 3.8, in the complete case. Their proofs also hold in the noncomplete case.

7. Caccioppoli inequalities

In this section Caccioppoli inequalities are proved, and in particular the logarithmic Caccioppoli inequality is studied. We start with an estimate for quasisubminimizers.

Proposition 7.1. *Let $u \ge 0$ be a Q-quasisubminimizer in Ω. Then for all nonnegative $\eta \in \operatorname{Lip}_b(\Omega)$,*

$$\int_\Omega g_u^p \eta^p \, d\mu \le c \int_\Omega u^p g_\eta^p \, d\mu,$$

where c only depends on p and Q.

This estimate was proved for unweighted $\mathbf{R}^n$ by Tolksdorf [40], Theorem 1.4, and for complete metric spaces in A. Björn [4], Theorem 4.1. The proof given in [4] (which was an easy adaptation of Tolksdorf's proof) applies also to the noncomplete case.

Proposition 7.2. *Let $u \ge 0$ be a Q-quasisubminimizer in Ω and $\alpha \ge 0$. Then for all nonnegative $\eta \in \operatorname{Lip}_b(\Omega)$,*

$$\int_\Omega u^\alpha g_u^p \eta^p \, d\mu \le C \int_\Omega u^{p+\alpha} g_\eta^p \, d\mu,$$

where C only depends on p and Q.

Proof. By Lemma 6.4, $(u - t^{1/\alpha})_+$ is also a Q-quasisubminimizer. Using Proposition 7.1 we see that

$$\begin{aligned}
\int_\Omega u^\alpha g_u^p \eta^p \, d\mu &= \int_0^\infty \int_{u^\alpha > t} g_u^p \eta^p \, d\mu \, dt \\
&= \int_0^\infty \int_{u > t^{1/\alpha}} g_{u - t^{1/\alpha}}^p \eta^p \, d\mu \, dt \\
&\le C \int_0^\infty \int_{u > t^{1/\alpha}} (u - t^{1/\alpha})^p g_\eta^p \, d\mu \, dt \\
&= C \int_\Omega \int_0^{u^\alpha} (u - t^{1/\alpha})^p \, dt \, g_\eta^p \, d\mu \\
&\le C \int_\Omega u^{p+\alpha} g_\eta^p \, d\mu.
\end{aligned}$$

□

The constant C is the same as in Proposition 7.1. A better estimate in the last step will give a better estimate of C, and in particular it is possible to show that $C \to 0$, as $\alpha \to \infty$, if we allow C to depend also on α.

Proposition 7.3. *Let $u > 0$ be a Q-quasisuperminimizer in Ω and $\alpha > 0$. Then for all nonnegative $\eta \in \operatorname{Lip}_b(\Omega)$,*

$$\int_\Omega u^{-\alpha-p} g_u^p \eta^p \, d\mu \le C \frac{\alpha + p}{\alpha} \int_\Omega u^{-\alpha} g_\eta^p \, d\mu,$$

where C only depends on p and Q.

In fact the constant C is the constant in Proposition 7.1.

Proof. Let first $M > 0$ be arbitrary and $v = (M - u)_+$. Then v is a Q-quasisubminimizer, by Lemma 6.4, and

$$g_v(x) = \begin{cases} g_u(x), & \text{if } u(x) < M, \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 7.1 (with C being the constant from there), we get

$$\int_{u<M} g_u^p \eta^p \, d\mu = \int_\Omega g_v^p \eta^p \, d\mu \le C \int_\Omega v^p g_\eta^p \, d\mu \le C M^p \int_{u<M} g_\eta^p \, d\mu.$$

We thus get

$$\begin{aligned}\int_\Omega u^{-\alpha-p} g_u^p \eta^p \, d\mu &= \int_0^\infty \int_{u^{-(\alpha+p)}>t} g_u^p \eta^p \, d\mu \, dt \\ &= \int_0^\infty \int_{u<t^{-1/(\alpha+p)}} g_u^p \eta^p \, d\mu \, dt \\ &\le C \int_0^\infty t^{-p/(\alpha+p)} \int_{u<t^{-1/(\alpha+p)}} g_\eta^p \, d\mu \, dt \\ &= C \int_\Omega \int_0^{u^{-(\alpha+p)}} t^{-p/(\alpha+p)} \, dt \, g_\eta^p \, d\mu \\ &= C \frac{\alpha+p}{\alpha} \int_\Omega u^{-\alpha} g_\eta^p \, d\mu. \qquad \square\end{aligned}$$

For superminimizers Proposition 7.3 can be improved.

Proposition 7.4. *Suppose $u > 0$ is a superminimizer in Ω and let $\alpha > 0$. Then for all nonnegative $\eta \in \operatorname{Lip}_b(\Omega)$,*

$$\int_\Omega u^{-\alpha-1} g_u^p \eta^p \, d\mu \le \Big(\frac{p}{\alpha}\Big)^p \int_\Omega u^{p-\alpha-1} g_\eta^p \, d\mu. \tag{7.1}$$

This result was proved in Kinnunen–Martio [28], Lemma 3.1, using a suitable test function and a convexity argument. Unfortunately, it does not seem possible to adapt their proof to quasisuperminimizers. In [28] the space was supposed to be complete, however, the proof can be easily modified in the noncomplete case. (Kinnunen–Martio had at their disposal regularity results saying that u is locally bounded away from 0; which may have been used implicitly in their proof. To clarify this point we note that using their argument we can obtain the corresponding inequality for $u_\delta := u + \delta$ for all $\delta > 0$, and from this the inequality for u is easily obtained using Fatou's lemma.)

For subminimizers Proposition 7.2 can be improved.

Proposition 7.5. *Suppose $u \ge 0$ is a subminimizer in Ω and let $\alpha > 0$. Then for all nonnegative $\eta \in \operatorname{Lip}_b(\Omega)$,*

$$\int_\Omega u^{\alpha-1} g_u^p \eta^p \, d\mu \le c \int_\Omega u^{p+\alpha-1} g_\eta^p \, d\mu,$$

where $c = (p/\alpha)^p$.

In Marola [32], this was proved under four additional assumptions, that X is complete, that u is locally bounded, that $\operatorname{ess\,inf}_\Omega u > 0$ and that $0 \le \eta \le 1$. The latter two are easy to remove by a limiting argument and a scaling, respectively. Moreover, the proof in [32] can be easily modified in the noncomplete case.

As for local boundedness, we show in Corollary 8.3 that every quasisubminimizer is locally bounded above, so assuming that u is locally bounded is no extra assumption. Note that we will not use Proposition 7.5 to obtain Corollary 8.3 (nor any other result in this paper). Here we just wanted to quote Proposition 7.5, as it may be of independent interest.

The following lemma is the logarithmic Caccioppoli inequality for superminimizers and it will play a crucial role in the proof of Harnack's inequality using Moser's method. We have not been able to prove a similar estimate for quasisuperminimizers. Proposition 7.6 was originally proved in Kinnunen–Martio [28].

Proposition 7.6. *Suppose that $u > 0$ is a superminimizer in Ω which is locally bounded away from 0. Let $v = \log u$. Then $v \in N^{1,p}_{\rm loc}(\Omega)$ and $g_v = g_u/u$ μ-a.e. in Ω. Furthermore, for every ball $B(z,r)$ with $B(z,2r) \subset \Omega$ we have*

$$\fint_{B(z,r)} g_v^p \, d\mu \le \frac{C}{r^p},$$

where $C = C_\mu(2p/(p-1))^p$.

The assumption that u is locally bounded away from 0 can actually be omitted, since this follows from Theorem 9.2, for the proof of which we however need this lemma in its present form.

Since we work in a possibly noncomplete metric space, there are really two possibilities for what "locally" may mean; either that for every $x \in \Omega$ there is a ball $B(x,r) \subset \Omega$, such that u is bounded in $B(x,r)$, or for every open set $G \Subset \Omega$, u is bounded in G (or equivalently every set $G \Subset \Omega$). For us the latter definition will be preferable.

We say that u is *locally bounded* in an open set Ω, if it is bounded in every open set $G \Subset \Omega$; locally bounded above and below are defined similarly.

Note also that the definition of locally here is in accordance with the definition of locally in $N^{1,p}_{\rm loc}$ given in Section 6.

Proof. Let $B(z,r)$ be a ball such that $B(z,2r) \subset \Omega$. As v is bounded below in $B(z,r)$ we have $v \in L^p(B(z,r))$. Clearly $g_v \le g_u/u$ μ-a.e. in Ω. We obtain the reverse inequality if we set $u = \exp v$, hence $g_v = g_u/u$ μ-a.e. in Ω. It follows that $g_v \in L^p_{\rm loc}(\Omega)$ and consequently that $v \in N^{1,p}_{\rm loc}(\Omega)$.

Let $\eta \in {\rm Lip}_b(B(z,2r))$ so that $0 \le \eta \le 1$, $\eta = 1$ on $B(z,r)$ and $g_\eta \le 2/r$. If we choose $\alpha = p-1$ in Proposition 7.4 we have

$$\int_\Omega g_v^p \eta^p \, d\mu = \int_\Omega u^{-p} g_u^p \eta^p \, d\mu \le C \int_\Omega g_\eta^p \, d\mu,$$

where $C = (p/(p-1))^p$. From this and the doubling property of μ we obtain

$$\int_{B(z,r)} g_v^p \, d\mu \le C r^{-p} \mu(B(z,r)),$$

where C is as in the statement of the lemma. $\square$

It is noteworthy that the lemma can be proved without applying Proposition 7.4. Namely, we obtain the desired result by choosing φ in the definition of superminimizers as $\varphi = \eta^p u^{1-p}$ and using a convexity argument as in the proof of Lemma 3.1 in Kinnunen–Martio [28].

Let us also note that in fact we have not used the Poincaré inequality to obtain any of the Caccioppoli inequalities in this section, with one exception. In order not to require that u is locally bounded in Proposition 7.5 we need to use Corollary 8.3. So if we add the assumption in Proposition 7.5 that u is locally bounded, then all the results in this section hold without assuming a Poincaré inequality.

Note also that our argument for obtaining Proposition 7.6 without the assumption that u is locally bounded away from 0 does require the Poincaré inequality.

8. Weak Harnack inequalities

In this section we prove weak Harnack inequalities for Q-quasisubminimizers (Theorem 8.2) and Q-quasisuperminimizers (Theorem 8.5).

We start with a technical lemma.

Lemma 8.1. *Let $\varphi(t)$ be a bounded nonnegative function defined on the interval $[a, b]$, where $0 \le a < b$. Suppose that for any $a \le t < s \le b$, φ satisfies*

$$\varphi(t) \le \theta\varphi(s) + \frac{A}{(s-t)^\alpha}, \tag{8.1}$$

where $\theta < 1$, A and α are nonnegative constants. Then

$$\varphi(\rho) \le C\frac{A}{(R-\rho)^\alpha}, \tag{8.2}$$

for all $a \le \rho < R \le b$, where C only depends on α and θ.

We refer to Giaquinta [12], Lemma 3.1, p. 161, for the proof. This lemma says that, under certain assumptions, we can get rid of the term $\theta\varphi(s)$.

The Moser iteration technique yields that nonnegative Q-quasisubminimizers are locally bounded.

Theorem 8.2. *Suppose that u is a nonnegative Q-quasisubminimizer in Ω. Then for every ball $B(z, r)$ with $B(z, 2r) \subset \Omega$ and any $q > 0$ we have*

$$\operatorname*{ess\,sup}_{B(z,r)} u \le C\left(\fint_{B(z,2r)} u^q \, d\mu\right)^{1/q}, \tag{8.3}$$

where C only depends on p, q, Q, C_μ and the constants in the weak Poincaré inequality.

Corollary 8.3. *Let u be a quasisubminimizer in Ω, then u is essentially locally bounded from above in Ω. Similarly any quasisuperminimizer in Ω is essentially locally bounded from below in Ω.*

Recall that we defined what is meant by locally bounded right after stating Proposition 7.6.

Proof. By Lemma 6.4, u_+ is a nonnegative quasisubminimizer. Let $G \Subset \Omega$ and let $\delta = \frac{1}{3}\operatorname{dist}(G, X \setminus \Omega)$. Using that X is a doubling space we can find a finite cover of G by balls $B_j = B(x_j, \delta)$, $x_j \in G$. By Theorem 8.2,

$$\operatorname*{ess\,sup}_{B_j} u \le \operatorname*{ess\,sup}_{B_j} u_+ \le C\biggl(\fint_{B(x_j,2\delta)} u_+^q \, d\mu\biggr)^{1/q} < \infty.$$

Since the cover is finite we see that $\operatorname{ess\,sup}_G u < \infty$. □

Proof of Theorem 8.2. First assume that $r \le \frac{1}{6}\operatorname{diam} X$ (which, of course, is immediate if X is unbounded).

Second we assume that $q \ge p$. Write $B_l = B(z, r_l), r_l = (1 + 2^{-l})r$ for $l = 0, 1, 2, ...,$ thus, $B(z, 2r) = B_0 \supset B_1 \supset \dots$. Let $\eta_l \in \operatorname{Lip}_b(B_l)$ so that $0 \le \eta_l \le 1$, $\eta_l = 1$ on $\overline{B}_{l+1}$ and $g_{\eta_l} \le 4 \cdot 2^l/r$ (choose, e.g., $\eta_l(x) = \min\{2(r_l - d(x, z))/(r_l - r_{l+1}) - 1, 1\}_+$). Fix $1 \le t < \infty$ and let

$$w_l = \eta_l u^{1+(t-1)/p} = \eta_l u^{\tau/p},$$

where $\tau := p + t - 1$. Then we have

$$g_{w_l} \le g_{\eta_l} u^{\tau/p} + \frac{\tau}{p} u^{(t-1)/p} g_u \eta_l \quad \mu\text{-a.e. in } \Omega,$$

and consequently

$$g_{w_l}^p \le 2^{p-1} g_{\eta_l}^p u^\tau + 2^{p-1}\biggl(\frac{\tau}{p}\biggr)^p u^{t-1} g_u^p \eta_l^p \quad \mu\text{-a.e. in } \Omega.$$

Using the Caccioppoli inequality, Proposition 7.2, with $\alpha = t - 1$ we obtain

$$\begin{aligned}\biggl(\fint_{B_l} g_{w_l}^p \, d\mu\biggr)^{1/p} &\le 2^{(p-1)/p}\biggl(\fint_{B_l}\biggl(g_{\eta_l}^p u^\tau + \biggl(\frac{\tau}{p}\biggr)^p u^{t-1} g_u^p \eta_l^p\biggr) d\mu\biggr)^{1/p} \\ &\le C\tau\biggl(\fint_{B_l} g_{\eta_l}^p u^\tau \, d\mu\biggr)^{1/p} \le C\tau\frac{2^l}{r}\biggl(\fint_{B_l} u^\tau \, d\mu\biggr)^{1/p},\end{aligned}$$

The Sobolev inequality (5.2) implies (here we use that $r_l \le 2r \le \frac{1}{3}\operatorname{diam} X$)

$$\begin{aligned}\biggl(\fint_{B_l} w_l^{\kappa p} \, d\mu\biggr)^{1/\kappa p} &\le C r_l \biggl(\fint_{B_l} g_{w_l}^p \, d\mu\biggr)^{1/p} \\ &\le C\tau(1 + 2^{-l}) r \frac{2^l}{r}\biggl(\fint_{B_l} u^\tau \, d\mu\biggr)^{1/p} \le C\tau 2^l \biggl(\fint_{B_l} u^\tau \, d\mu\biggr)^{1/p}\end{aligned}$$

Using the doubling property of μ we have (remember that $w_l = u^{\tau/p}$ on B_{l+1})

$$\left(\fint_{B_{l+1}} (u^{\tau/p})^{\kappa p}\, d\mu\right)^{1/\kappa p} \le C\tau 2^l \left(\fint_{B_l} u^\tau\, d\mu\right)^{1/p}.$$

Hence, we obtain

$$\left(\fint_{B_{l+1}} u^{\kappa\tau}\, d\mu\right)^{1/\kappa\tau} \le (C\tau 2^l)^{p/\tau} \left(\fint_{B_l} u^\tau\, d\mu\right)^{1/\tau}.$$

This estimate holds for all $\tau \ge p$. We use it with $\tau = q\kappa^l$ to obtain

$$\left(\fint_{B_{l+1}} u^{q\kappa^{l+1}}\, d\mu\right)^{1/q\kappa^{l+1}} \le (Cq2^l\kappa^l)^{p/q\kappa^l} \left(\fint_{B_l} u^{q\kappa^l}\, d\mu\right)^{1/q\kappa^l}.$$

By iterating we obtain the desired estimate

$$\begin{aligned}
\operatorname*{ess\,sup}_{B(z,r)} u &\le \left((Cq)^{\sum_{i=0}^\infty \kappa^{-i}} (2\kappa)^{\sum_{i=0}^\infty i\kappa^{-i}}\right)^{p/q} \left(\fint_{B(z,2r)} u^q\, d\mu\right)^{1/q} \\
&= \left((Cq)^{\kappa/(\kappa-1)} (2\kappa)^{\kappa/(\kappa-1)^2}\right)^{p/q} \left(\fint_{B(z,2r)} u^q\, d\mu\right)^{1/q} \\
&\le C\left(\fint_{B(z,2r)} u^q\, d\mu\right)^{1/q}. \qquad (8.4)
\end{aligned}$$

The theorem is proved for $q \ge p$ and $r \le \frac{1}{6} \operatorname{diam} X$.

By the doubling property of the measure and (2.1), it is easy to see that (8.4) can be reformulated in a bit different manner. Namely, if $0 \le \rho < \tilde{r} \le 2r$, then

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \le \frac{C}{(1-\rho/\tilde{r})^{s/q}} \left(\fint_{B(z,\tilde{r})} u^q\, d\mu\right)^{1/q}. \qquad (8.5)$$

See Kinnunen–Shanmugalingam [29], Remark 4.4.

If $0 < q < p$ we want to prove that

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \le \frac{C}{(1-\rho/2r)^{s/q}} \left(\fint_{B(z,2r)} u^q\, d\mu\right)^{1/q},$$

when $0 \le \rho < 2r < \infty$. Now suppose that $0 < q < p$ and let $0 \le \rho < \tilde{r} \le 2r$. We choose $q = p$ in (8.5), then

$$\begin{aligned}
\operatorname*{ess\,sup}_{B(z,\rho)} u &\le \frac{C}{(1-\rho/\tilde{r})^{s/p}} \left(\fint_{B(z,\tilde{r})} u^q u^{p-q}\, d\mu\right)^{1/p} \\
&\le \frac{C}{(1-\rho/\tilde{r})^{s/p}} \left(\operatorname*{ess\,sup}_{B(z,\tilde{r})} u\right)^{1-q/p} \left(\fint_{B(z,\tilde{r})} u^q\, d\mu\right)^{1/p}
\end{aligned}$$

By Young's inequality

$$\begin{aligned}\operatorname*{ess\,sup}_{B(z,\rho)} u &\le \frac{p-q}{p} \operatorname*{ess\,sup}_{B(z,\tilde r)} u + \frac{C}{(1-\rho/\tilde r)^{s/q}} \biggl(\fint_{B(z,\tilde r)} u^q \, d\mu \biggr)^{1/q} \\ &\le \frac{p-q}{p} \operatorname*{ess\,sup}_{B(z,\tilde r)} u + \frac{C}{(\tilde r-\rho)^{s/q}} \biggl((2r)^s \fint_{B(z,2r)} u^q \, d\mu \biggr)^{1/q},\end{aligned}$$

where the doubling property (2.1) was used to obtain the last inequality. We need to get rid of the first term on the right-hand side. By Lemma 8.1 (let $\varphi(t) = \operatorname{ess\,sup}_{B(z,t)} u$) we have

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \le \frac{C}{(1-\rho/2r)^{s/q}} \biggl(\fint_{B(z,2r)} u^q \, d\mu \biggr)^{1/q}$$

for all $0 \le \rho < 2r$. If we set $\rho = r$, we obtain (8.3) for every $0 < q < p$ and the proof is complete for the case when $r \le \frac{1}{6} \operatorname{diam} X$.

Assume now that $r > \frac{1}{6} \operatorname{diam} X$ and let $r' = \frac{1}{12} \operatorname{diam} X$. Then we can find $z' \in B(z,r)$ such that

$$\operatorname*{ess\,sup}_{B(z',r')} u \ge \operatorname*{ess\,sup}_{B(z,r)} u.$$

Using the doubling property and the fact that $B(z',2r') \subset B(z,2r) \subset X = B(z',12r')$ we find that

$$\operatorname*{ess\,sup}_{B(z,r)} u \le \operatorname*{ess\,sup}_{B(z',r')} u \le C \biggl(\fint_{B(z',2r')} u^q \, d\mu \biggr)^{1/q} \le C \biggl(\fint_{B(z,2r)} u^q \, d\mu \biggr)^{1/q},$$

which makes the proof complete. □

Remark 8.4. The quasi(sub)minimizing property (6.1) is not needed in the proof of Theorem 8.2. As our proof shows, it is enough to have a Caccioppoli inequality like in Proposition 7.2.

Next we present a certain reverse Hölder inequality for Q-quasisuperminimizers.

Theorem 8.5. *Suppose that u is a nonnegative Q-quasisuperminimizer in Ω. Then for every ball $B(z,r)$ with $B(z,2r) \subset \Omega$ and any $q > 0$ we have*

$$\operatorname*{ess\,inf}_{B(z,r)} u \ge C \biggl(\fint_{B(z,2r)} u^{-q} \, d\mu \biggr)^{-1/q}, \tag{8.6}$$

where C only depends on p, q, Q, C_μ and the constants in the weak Poincaré inequality.

Proof. The result can be obtained for general r after having obtained it for $r \le \frac{1}{6} \operatorname{diam} X$ in the same way as in the proof of Theorem 8.2. We may thus assume that $r \le \frac{1}{6} \operatorname{diam} X$.

Assume next that $u > 0$. As in the proof of Theorem 8.2, write $B_l = B(z, r_l), r_l = (1+2^{-l})r$ for $l = 0, 1, 2, \ldots$. Let $\eta_l \in \mathrm{Lip}_b(B_l)$ so that $0 \le \eta_l \le 1$, $\eta_l = 1$ on $\overline{B}_{l+1}$ and $g_{\eta_l} \le 4 \cdot 2^l/r$. Fix $t \ge \max\{1, q+p-1\}$. Thus $\tau := t + 1 - p \ge q$. Let

$$w_l = \eta_l u^{1+(-t-1)/p} = \eta_l u^{-\tau/p}.$$

Then we have

$$g_{w_l} \le g_{\eta_l} u^{-\tau/p} + \left(\frac{\tau}{p}\right) u^{(-t-1)/p} g_u \eta_l \quad \mu\text{-a.e. in } \Omega$$

and consequently

$$g_{w_l}^p \le 2^{p-1} g_{\eta_l}^p u^{-\tau} + 2^{p-1} \left(\frac{\tau}{p}\right)^p u^{-t-1} g_u^p \eta_l^p \quad \mu\text{-a.e. in } \Omega.$$

Using the Caccioppoli inequality, Proposition 7.3, with $\alpha = \tau$, we obtain

$$\begin{aligned} \left(\fint_{B_l} g_{w_l}^p \, d\mu\right)^{1/p} &\le 2^{(p-1)/p} \left(\fint_{B_l} \left(g_{\eta_l}^p u^{-\tau} + \left(\frac{\tau}{p}\right)^p u^{-t-1} g_u^p \eta_l^p\right) d\mu\right)^{1/p} \\ &\le C\tau \left(\fint_{B_l} g_{\eta_l}^p u^{-\tau} \, d\mu\right)^{1/p} \le C\tau \frac{2^l}{r} \left(\fint_{B_l} u^{-\tau} \, d\mu\right)^{1/p}, \end{aligned}$$

where we note that C depends on q but not on τ. The Sobolev inequality (5.2) implies

$$\begin{aligned} \left(\fint_{B_l} w_l^{\kappa p} \, d\mu\right)^{1/\kappa p} &\le C r_l \left(\fint_{B_l} g_{w_l}^p \, d\mu\right)^{1/p} \\ &\le C\tau (1+2^{-l}) r \frac{2^l}{r} \left(\fint_{B_l} u^{-\tau} \, d\mu\right)^{1/p} \\ &\le C\tau 2^l \left(\fint_{B_l} u^{-\tau} \, d\mu\right)^{1/p} \end{aligned}$$

Using the doubling property of μ we have (notice that $w_l = u^{-\tau/p}$ on B_{l+1})

$$\left(\fint_{B_{l+1}} (u^{-\tau/p})^{\kappa p} \, d\mu\right)^{1/\kappa p} \le C\tau 2^l \left(\fint_{B_l} u^{-\tau} \, d\mu\right)^{1/p}.$$

Hence, we obtain

$$\left(\fint_{B_{l+1}} u^{-\kappa\tau} \, d\mu\right)^{-1/\kappa\tau} \ge (C\tau 2^l)^{-p/\tau} \left(\fint_{B_l} u^{-\tau} \, d\mu\right)^{-1/\tau}.$$

This estimate holds for all $\tau > 0$. We use it with $\tau = q\kappa^l$ to obtain

$$\left(\fint_{B_{l+1}} u^{-q\kappa^{l+1}} \, d\mu\right)^{-1/q\kappa^{l+1}} \ge (Cq 2^l \kappa^l)^{-p/q\kappa^l} \left(\fint_{B_l} u^{-q\kappa^l} \, d\mu\right)^{-1/q\kappa^l}.$$

By iterating as in the proof of Theorem 8.2, we obtain the desired estimate

$$\operatorname*{ess\,inf}_{B(z,r)} u \ge C\biggl(\fint_{B(z,2r)} u^{-q}\,d\mu\biggr)^{-1/q}.$$

The proof is complete for $u>0$.

If u is a nonnegative Q-quasisuperminimizer in Ω, it is evident that also $u+\beta$ is for all constants $\beta>0$. Hence we may apply (8.6) to obtain

$$\operatorname*{ess\,inf}_{B(z,r)} (u+\beta) \ge C\biggl(\fint_{B(z,2r)} (u+\beta)^{-q}\,d\mu\biggr)^{-1/q}$$

for all $\beta>0$, where the constant C is independent of β. Letting $\beta\to 0+$ completes the proof. □

Remark 8.6. As in the proof of Theorem 8.2 the quasi(super)minimizing property (6.1) is not really needed. Again, it is enough to have a Caccioppoli inequality in the spirit of Proposition 7.3.

9. Harnack's inequality for minimizers

We stress that results in this section are valid only for (super)minimizers of the p-Dirichlet integral.

A locally integrable function u in Ω is said to belong to $\mathrm{BMO}(\Omega)$ if the inequality

$$\fint_B |u-u_B|\,d\mu \le C \tag{9.1}$$

holds for all balls $B\subset\Omega$. The smallest bound C for which (9.1) is satisfied is said to be the "BMO-norm" of u in this space, and is denoted by $\|u\|_{\mathrm{BMO}(\Omega)}$.

We will need the following result.

Theorem 9.1. *Let $u\in\mathrm{BMO}(B(x,2r))$ and let $q=1/6C_\mu\|u\|_{\mathrm{BMO}(\Omega)}$, then*

$$\fint_{B(x,r)} e^{q|u-u_B|}\,d\mu \le 16.$$

This theorem was proved in Buckley [9], Theorem 2.2. The proof is related to the proof of the John–Nirenberg inequality, and in fact this theorem can be obtained as a rather straightforward corollary of the John–Nirenberg inequality.

For the formulation of the John–Nirenberg inequality and proofs of it valid in doubling metric spaces we refer to [9] and the appendix in Mateu–Mattila–Nicolau–Orobitg [33].

Now we are ready to provide the proof for the weak Harnack inequality. A sharp version of the following theorem is proved in Kinnunen–Martio [28], under the additional assumption that the space is complete.

Theorem 9.2. *If u is a nonnegative superminimizer in Ω, then there are $q>0$ and $C>0$, only depending on p, C_μ and the constants in the weak Poincaré inequality, such that*

$$\left(\fint_{B(z,2r)} u^q\,d\mu\right)^{1/q} \le C \operatorname*{ess\,inf}_{B(z,r)} u \tag{9.2}$$

for every ball $B(z,r)$ such that $B(z,20\lambda r)\subset\Omega$.

Here λ is the dilation constant in the weak Poincaré inequality.

Proof. Let $u>0$ be bounded away from 0. By Theorem 8.5 we have

$$\begin{aligned}\operatorname*{ess\,inf}_{B(z,r)} u &\ge C\left(\fint_{B(z,2r)} u^{-q}\,d\mu\right)^{-1/q}\\ &= C\left(\fint_{B(z,2r)} u^{-q}\,d\mu \fint_{B(z,2r)} u^q\,d\mu\right)^{-1/q}\left(\fint_{B(z,2r)} u^q\,d\mu\right)^{1/q}.\end{aligned}$$

To complete the proof, we have to show that

$$\fint_{B(z,2r)} u^{-q}\,d\mu \fint_{B(z,2r)} u^q\,d\mu \le C$$

for some $q>0$. Write $v=\log u$. We want to show that $v\in \mathrm{BMO}(B(z,4r))$. Let $B(x,r')\subset B(z,4r)$ and let $r''=\min\{8r,r'\}$. It is easy to see that $B(x,r')=B(x,r'')$ (recall that in metric spaces balls may not have unique centre or radius). It is also easy to see that

$$2B(x,\lambda r'')\subset B(z,16\lambda r+d(x,z))\subset B(z,20\lambda r)\subset\Omega.$$

By the weak $(1,p)$-Poincaré inequality and Proposition 7.6 we have

$$\begin{aligned}\fint_{B(x,r')} |v-v_{B(x,r')}|\,d\mu &= \fint_{B(x,r'')} |v-v_{B(x,r'')}|\,d\mu\\ &\le Cr\left(\fint_{B(x,\lambda r'')} g_v^p\,d\mu\right)^{1/p} \le C',\end{aligned}$$

where C' only depends on p, C_μ and the constants in the weak Poincaré inequality. Thus $\|v\|_{\mathrm{BMO}(B(z,4r))}\le C'$.

Let now $q=1/6C'C_\mu$. By Theorem 9.1,

$$\begin{aligned}\fint_{B(z,2r)} e^{-qv}\,d\mu \fint_{B(z,2r)} e^{qv}\,d\mu &= \fint_{B(z,2r)} e^{q(v_{B(z,2r)}-v)}\,d\mu \fint_{B(z,2r)} e^{q(v-v_{B(z,2r)})}\,d\mu\\ &\le \left(\fint_{B(z,2r)} e^{q|v-v_{B(z,2r)}|}\,d\mu\right)^2 \le 256,\end{aligned}$$

from which the claim follows for u bounded away from 0.

If u is an arbitrary nonnegative superminimizer, then clearly $u_\beta := u+\beta\ge\beta$ is a superminimizer for all constants $\beta>0$. Hence we may apply (9.2) to u_β. Letting $\beta\to 0+$ and using Fatou's lemma completes the proof. □

From this we easily obtain Harnack's inequality.

Theorem 9.3. *Suppose that u is a nonnegative minimizer in Ω. Then there exists a constant $C \geq 1$, only depending on p, C_μ and the constants in the weak Poincaré inequality, such that*

$$\operatorname*{ess\,sup}_{B(z,r)} u \leq C \operatorname*{ess\,inf}_{B(z,r)} u$$

for every ball $B(z,r)$ for which $B(z,20\lambda r) \subset \Omega$.

Here λ is the dilation constant in the weak Poincaré inequality.

Proof. Combine Theorems 8.2 and 9.2. □

From Harnack's inequality it follows that minimizers are locally Hölder continuous (after modification on a set of measure zero) and satisfy the strong maximum principle, see, e.g., Giusti [15]. Furthermore, we obtain Liouville's theorem as a corollary of Harnack's inequality.

Corollary 9.4. (Liouville's theorem) *If u is a bounded or nonnegative p-harmonic function on all of X, then u is constant.*

By definition, a *p-harmonic function* is a continuous minimizer.

Proof. Let $v = u - \inf_X u$. For $x \in X$ we thus get,

$$v(x) \leq \sup_{B(x,r)} v \leq C \inf_{B(x,r)} v \to 0, \quad \text{as } r \to \infty.$$

Thus $v \equiv 0$, and u is constant. □

10. The need for λ in Theorems 9.2 and 9.3

It may seem that a better proof could eliminate the need for λ in Theorem 9.2 and consequently also in Theorem 9.3, in particular after noting that no λ is needed in Theorems 8.2 and 8.5. However, λ is really essential in Theorems 9.2 and 9.3.

Example 10.1. Let $X_M = \mathbf{R}^2 \setminus ((-M,M) \times (0,1))$, $M \geq 1$, equipped with Euclidean distance and the restriction of Lebesgue measure, which is doubling. By, e.g., Theorem 10.5 in Hajłasz–Koskela [17], X_M supports a weak $(1,1)$-Poincaré inequality.

Let us fix $M \geq 2$ and let $X = X_M$. Let next $\Omega = (-M,M)^2 \cap X$ (which is disconnected) and

$$u(x,y) = \begin{cases} 1, & \text{if } y \geq 1, \\ 0, & \text{if } y \leq 0. \end{cases}$$

Since $g_u \equiv 0$ we see that u is p-harmonic in Ω (for all p). Let further $B = B((0,0),2)$ (as a ball in X). Then $\frac{1}{2}MB \subset \Omega$, and this shows that the constant 20λ cannot be replaced by $\frac{1}{2}M$ neither in Theorem 9.2 nor in Theorem 9.3. By varying M we see that the constant 20λ in Theorems 9.2 and 9.3 cannot be replaced by any absolute constant.

Note that in this example X is complete. However, Ω was disconnected. We next make a modification of Ω to obtain a connected counterexample as well.

Let $\Omega_\varepsilon = \Omega \cup \bigl(B((-M,0),M) \setminus \overline{B((-M,0),M-\varepsilon)}\bigr)$, where $0 < \varepsilon < 1$ and the balls are taken within X. Note that Ω_ε is a connected subset of X. Let next $f_\varepsilon(x,y) = \min\{y_+,1\}$ on $\partial\Omega_\varepsilon$ and let u_ε be the solution to the Dirichlet problem with boundary values f_ε on $\partial\Omega_\varepsilon$ for $p=2$, i.e. the 2-harmonic function which takes the boundary values q.e. (see, e.g., Björn–Björn–Shanmugalingam [6]).

The harmonic measure of $\partial\Omega \setminus \partial\Omega_\varepsilon$ with respect to Ω tends to 0. Hence $u_\varepsilon \to u$ uniformly on B, which shows that the constant 20λ cannot be replaced by $\frac{1}{2}M$ in Theorems 9.2 and 9.3, even if Ω is required to be connected.

We know that X_1 supports a weak Poincaré inequality with some dilation constant λ_1. By applying the affine map $(x,y) \mapsto (Mx,y)$ it is easy to see that X_M satisfies a weak Poincaré inequality with dilation constant $M\lambda_1$. This shows that the constant 20λ in Theorems 9.2 and 9.3 has the right growth.

This example also shows that the dilation constant from the Poincaré inequality (called τ' in [29]) needs to be inserted in the condition on the balls $B(z,R)$ in Corollary 7.3 in Kinnunen–Shanmugalingam [29]. The authors [30] have communicated that the first result in need of a slight modification in [29] is Theorem 5.2. Several of the results in the following sections need similar treatment.

The results and proofs of [29] have been referred to in several papers. It should be observed that all the qualitative results in [29], as well as in the papers depending on it, are not affected by this inadvertence. However, there are certain quantitative statements in Kinnunen–Martio [26], [27] and A. Björn [3] that need to be modified in a similar fashion.

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ISBN 951-22-7878-2
ISSN 0784-3143