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Abstract: *We prove Harnack's inequality for first eigenfunctions of the p-Laplacian in metric measure spaces. The proof is based on the famous Moser iteration method, which has the advantage that it only requires the $(1, p)$-Poincaré inequality. As a by-product we obtain the continuity and the fact that first eigenfunctions do not change sign in bounded domains.*

AMS subject classifications: 35P30, 35J20

Keywords: Sobolev space, Newtonian space, Caccioppoli estimate, Harnack's inequality, First eigenvalue, First eigenfunction, Rayleigh quotient, p-Dirichlet integral, Moser iteration

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1 Introduction

The eigenvalue problem of the p-Laplacian is to find functions $u \in W_0^{1,p}(\Omega)$ that satisfy the equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, \quad 1 < p < \infty, \tag{1.1}$$

for some $\lambda \neq 0$ in an open set $\Omega \subset \mathbf{R}^n$. This problem was apparently first studied by Thelin in [26]. The first eigenvalue is defined as the least real number λ for which the equation (1.1) has a non-trivial solution u. In defining eigenvalues we shall interpret equation (1.1) in the weak sense. The first eigenvalue $\lambda_1 = \lambda_1(\Omega)$ is obtained by minimizing the Rayleigh quotient

$$\lambda_1 = \inf_u \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} \tag{1.2}$$

with $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. The minimization problem (1.2) is equivalent to the corresponding Euler-Lagrange equation (1.1) with $\lambda = \lambda_1$.

In this note we consider first eigenfunctions, that is, solutions u of the eigenvalueproblem (1.2) on a metric measure space X by replacing the standard Sobolev space $W_0^{1,p}(\Omega)$ with the Newtonian space $N_0^{1,p}(\Omega)$. Since differential equations are problematic in metric measure spaces, we use only the variational approach. This has been previously studied in Pere [22], where the author proves that first eigenfunctions always exist in our setting and they have a locally Hölder continuous representative. The proof of the Hölder continuity in [22] is based on the famous De Giorgi method (see De Giorgi [2], Giaquinta [5] and Giusti [6]).

We continue the study of [22] by proving that first eigenfunctions are bounded and non-negative first eigenfunctions satisfy Harnack's inequality. Our methods require that $\Omega \subset X$ is bounded. The proof of the boundedness is based on a method by Ladyzhenskaya and Uraltseva [17]. The proof of the Harnack's inequality uses the Moser iteration technique (see Moser [20] and [21]), which was adapted to the metric setting in Marola [19]. More general functionals of the calculus of variations are studied in Giusti [6]. However, we obtain different Harnack's inequality than in [6] when we confine Giusti's setting to our case, see Theorem 7.10 in [6] and Theorem 5.5 in Section 5. We also give a simple proof for the continuity of eigenfunctions by combining the weak Harnack estimates of the two different methods. Observe here that continuity does not easily follow from Harnack's inequality since the sum of an eigenfunction and a constant is not an eigenfunction in general.

The advantage of our methods is that they work under weaker assumptions than those of Pere [22]. The difference is that we only require the weak $(1,p)$-Poincare inequality instead of a weak $(1,q)$-Poincaré inequality for some $1 < q < p$. The latter inequality appears as a basic assumption on several papers dealing with nonlinear potential theory in metric spaces. In fact, the weak $(1, q)$-Poincaré inequality is crucial for the latter part of the De Giorgi method which builds up the local Hölder continuity.

The work is organized as follows. In the preliminary section we focus on notation, definitions and concepts which appear in this work. In the third section we prove that first eigenfunctions are bounded. The main results of this paper are included in Section 4, where we establish a Caccioppoli type estimate and weak Harnack estimates. The final section 5 discusses Harnack's inequality and continuity.

2 Preliminaries

We assume that X is a metric measure space equipped with a Borel regular measure μ. We assume that the measure of every nonempty open set is positive and that the measure of every bounded set is finite. The further requirements on the space and the measure are included in the end of this section. Throughout, $B_r := B(z, r)$ refers to an open ball with the center z and radius $r > 0$. Constants are usually labeled as c, and their values may change even in a single line. If A is a subset of X, then χ_A denotes the characteristic function of A. We let $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$. If not otherwise stated, p is a real number satisfying $1 < p < \infty$.

By a path in X we mean any continuous mapping $\gamma : [a, b] \to X$, where $[a, b], a < b$, is an interval in $\mathbf{R}$. Its image will be denoted by $|\gamma| = \gamma([a, b])$ and $l(\gamma)$ denotes the length of γ. We say that the curve is rectifiable if $l(\gamma) < \infty$. The collection of all non-constant rectifiable paths $\gamma : [a, b] \to X$ is denoted by Γ_{rect}. Throughout the paper we will assume that every path is nonconstant, compact and rectifiable. A path can thus be parametrized by its arc length. See Heinonen [10], Heinonen–Koskela [11] and Väisälä [27] for the discussion of rectifiable paths and path integration.

The p-modulus of a family of paths Γ in X is the number

$$\mathrm{Mod}_p(\Gamma) = \inf_\rho \int_X \rho^p \, d\mu,$$

where the infinum is taken over all non-negative Borel measurable functions ρ so that

$$\int_\gamma \rho \, ds \geq 1$$

for all rectifiable paths γ belonging to Γ. It is well-known that the p-modulus is an outer measure on the collection of all paths in X. From the above definition it is clear that the p-modulus of the family of all non-rectifiable paths is zero, thus non-rectifiable paths are not interesting in this study. See Fuglede [3], [10] and [27] for additional information on p-modulus.

In a metric measure space an upper gradient is a counterpart for the Sobolev gradient.

Definition 2.1. Let u be an extended real-valued function on X. We say that a non-negative Borel measurable function g is an *upper gradient* of u if for all rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds. \tag{2.2}$$

See Cheeger [1], [10], [11] and Shanmugalingam [24] for a discussion on upper gradients. A property is said to hold for *p-almost all* paths, if the set of paths for which the property fails is of zero p-modulus. If (2.2) holds for p-almost all paths γ, then g is said to be a *p-weak upper gradient* of u. It is known that if $1 < p < \infty$ and u has a p-weak upper gradient in $L^p(X)$ then u has the least p-weak upper gradient g_u in $L^p(X)$. It is the smallest in the sense that if g is another p-weak upper gradient in $L^p(X)$ of u then $g \geq g_u$ μ-almost everywhere. This fact has been proved in Shanmugalingam [23]. An alternative proof is given in [1].

Newtonian spaces

Here we introduce the notion of Sobolev spaces on a metric measure space based on the concept of upper gradients. Following Shanmugalingam [24] we define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all real-valued p-integrable functions u on X that have a p-integrable p-weak upper gradient g_u. We equip this space with a seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \Big(\|u\|^p_{L^p(X)} + \|g_u\|^p_{L^p(X)} \Big)^{1/p},$$

and say that u and v belong to the same equivalence class, write $u \sim v$, if $\|u - v\|_{\widetilde{N}^{1,p}(X)} = 0$. The *Newtonian space* $N^{1,p}(X)$ is defined to be the space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\widetilde{N}^{1,p}(X)}.$$

For basic properties of Newtonian space we refer to [24].

Definition 2.3. Let $u : X \to \mathbf{R}$ be a given function and $\gamma \in \Gamma_{\text{rect}}$ arc-length parametrized path in X. We say that

(i) u is *absolutely continuous along a path* γ if $u \circ \gamma$ is absolutely continuous on $[0, l(\gamma)]$,

(ii) u is *absolutely continuous on p-almost every curve*, or simply ACC_p, if for p-almost every γ, $u \circ \gamma$ is absolutely continuous.

It is very useful to know that if u is a function in $\widetilde{N}^{1,p}(X)$, then u is ACC_p. See [24] for the proof.

The *p-capacity* of a set $E \subset X$ with respect to the space $N^{1,p}(X)$ is defined by

$$\mathrm{Cap}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infinum is taken over all functions $u \in \widetilde{N}^{1,p}(X)$ whose restriction to E is bounded below by 1. Sets of zero capacity are also of measure zero, but the converse is not true. See Kinnunen–Martio [16] for more properties of the capacity in the metric setting.

In order to give a definition of first eigenfunctions we need a counterpart of the Sobolev functions with zero boundary values in a metric measure space. Let $\Omega \subset X$. Following the method of Kilpeläinen–Kinnunen–Martio [12], we define the space $\widetilde{N}^{1,p}_0(\Omega)$ to be the set of functions $\widetilde{u} \in \widetilde{N}^{1,p}(X)$ for which

$$\mathrm{Cap}_p\left(\{x \in X \setminus \Omega \ : \ \widetilde{u}(x) \neq 0\}\right) = 0.$$

The *Newtonian space with zero boundary values* $N^{1,p}_0(\Omega)$ is then $\widetilde{N}^{1,p}_0(\Omega)/\sim$ equipped with the norm

$$\|u\|_{N^{1,p}_0(\Omega)} = \|\widetilde{u}\|_{\widetilde{N}^{1,p}(X)}.$$

The norm on $N^{1,p}_0(\Omega)$ is unambiguously defined by Shanmugalingam [25] and the obtained space is a Banach space. Note also that if $\mathrm{Cap}_p(X \setminus \Omega) = 0$, then $N^{1,p}_0(\Omega) = N^{1,p}(X)$. In what follows, we usually identify the equivalence class with its representative.

The inequalities of Poincaré and Sobolev

We will impose some further requirements on the space and the measure. Namely, the measure μ is said to be *doubling* if there is a constant $c_\mu \geq 1$, called the *doubling constant* of μ, so that

$$\mu(B(z, 2r)) \leq c_\mu \mu(B(z, r)) \tag{2.4}$$

for every open ball $B(z, r)$ in X. By the doubling property, if $B(y, R)$ is a ball in X, $z \in B(y, R)$ and $0 < r \leq R < \infty$, then

$$\frac{\mu(B(z, r))}{\mu(B(y, R))} \geq c \left(\frac{r}{R}\right)^Q \tag{2.5}$$

for $c = c(c_\mu) > 0$ and $Q = \log_2 c_\mu$. The exponent Q serves as a counterpart of dimension related to the measure. A metric space X is said to be *doubling* if there exists a constant $c < \infty$ such that every ball $B(z, r)$ can be covered by at most c balls with the radius $r/2$. If X is equipped with a doubling measure, then X is doubling.

Let $1 < p < \infty$. The space X is said to *support a weak* $(1,p)$*-Poincaré inequality* if there are constants $c > 0$ and $\tau \geq 1$ such that

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu \leq cr \biggl(\fint_{B(z,\tau r)} g_u^p \, d\mu \biggr)^{1/p} \tag{2.6}$$

for all balls $B(z,r)$ in X, for all integrable functions u in $B(z,r)$ and for all p-weak upper gradients g_u of u. If $\tau = 1$, the space is said to support a $(1,p)$-Poincaré inequality. A result in Hajłasz–Koskela [7] (see also Hajłasz–Koskela [8]) shows that in a doubling measure space a weak $(1,p)$-Poincaré inequality implies a *Sobolev-Poincaré inequality.* More precisely, there is $c = c(p, \kappa, c_\mu) > 0$ such that

$$\biggl(\fint_{B(z,r)} |u - u_{B(z,r)}|^{\kappa p} \, d\mu \biggr)^{1/\kappa p} \leq cr \biggl(\fint_{B(z,5\tau r)} g_u^p \, d\mu \biggr)^{1/p}, \tag{2.7}$$

where $1 \leq \kappa \leq Q/(Q-p)$ if $1 < p < Q$ and $\kappa = 2$ if $p \geq Q$, for all balls $B(z,r)$ in X, for all integrable functions u in $B(z,r)$ and for all p-weak upper gradients g_u of u. We will also need an inequality for Newtonian functions with zero boundary values. If $u \in N_0^{1,p}((B(z,r))$, then there exists $c = c(p, c_\mu) > 0$ such that

$$\biggl(\fint_{B(z,r)} |u|^{\kappa p} \, d\mu \biggr)^{1/\kappa p} \leq cr \biggl(\fint_{B(z,r)} g_u^p \, d\mu \biggr)^{1/p}, \tag{2.8}$$

where $0 < 2r < \operatorname{diam}(X)$. For this result we refer to Kinnunen–Shanmugalingam [13]. In [13] the space was assumed to support a weak $(1,q)$-Poincaré inequality for some q with $1 < q < p$. However, this assumption is not needed in the proof of (2.8).

Minimizers and superminimizers

We next introduce the concept of p-(super)minimizer of the p-energy integral by following Kinnunen–Martio [14].

Definition 2.9. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-minimizer of the p-energy integral* in Ω if

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu \tag{2.10}$$

holds for all open $\Omega' \Subset \Omega$ for every $v \in N^{1,p}(\Omega')$ such that $v - u \in N_0^{1,p}(\Omega')$. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *p-superminimizer* in Ω if (2.10) holds for all open $\Omega' \Subset \Omega$ for every $v \in N^{1,p}(\Omega')$ such that $v - u \in N_0^{1,p}(\Omega')$ and $v \geq u$ μ-almost everywhere. A function is called a *p-subminimizer* if $-u$ is a p-superminimizer.

A function is a p-minimizer in Ω if, and only if, both u and $-u$ are p-superminimizer or it is both a p-superminimizer and p-subminimizer in Ω. Clearly if u is a p-superminimizer, then $\alpha u + \beta$ is a p-superminimizer when $\alpha \geq 0$ and $\beta \in \mathbf{R}$. Check Kinnunen–Martio [14, 15] for (super)minimizers in metric measure spaces.

Eigenfunctions

We define a first eigenfunction of the p-Laplacian using the variational approach. In the Euclidean space this definition is equivalent to equation (1.1) with $\lambda = \lambda_1$.

Definition 2.11. Let $\Omega \subset X$ be a bounded domain with $\mathrm{Cap}_p(X \setminus \Omega) > 0$ and $1 < p < \infty$. If $u \in N_0^{1,p}(\Omega)$, $u \neq 0$, minimizes the functional $J\colon N_0^{1,p}(\Omega) \to \mathbf{R}$,

$$J(v) = \frac{\int_\Omega g_v^p \, d\mu}{\int_\Omega |v|^p \, d\mu},$$

in $N_0^{1,p}(\Omega)$, then u is called a *first eigenfunction of the p-Laplacian* and $\lambda_1 = J(u) > 0$ is the corresponding *first eigenvalue*. In what follows we drop the subscript 1 from λ_1.

Remark 2.12. Observe that $N_0^{1,p}(\Omega) = N^{1,p}(X)$ if $\mathrm{Cap_p}(X \setminus \Omega) = 0$. If, in addition, $\mu(X) < \infty$, constant functions minimize the functional J and $\lambda = 0$ is the corresponding eigenvalue. Hence we have excluded this trivial case in Definition 2.11. If $\mathrm{Cap_p}(X \setminus \Omega) > 0$, we have an explicit lower bound for λ. Indeed, the Sobolev inequality (2.8) implies

$$\lambda \geq \frac{1}{c^p \operatorname{diam}(\Omega)^p},$$

where $c = c(p, c_\mu) > 0$ is the same constant as in (2.8).

The existence of first eigenfunctions is proved in Pere [22]. Note that the minimizers of the Rayleigh quotient also minimize the following functional, see [22].

Lemma 2.13. *Let $u \in N_0^{1,p}(\Omega)$ be a first eigenfunction of the p-Laplacian in Ω, and let λ be the corresponding eigenvalue. Then u minimizes the integral*

$$\hat{J}(v) = \int_\Omega (g_v^p - \lambda |v|^p) \, d\mu$$

in the set $N_0^{1,p}(\Omega)$.

We enclose this preliminary section by a simple lemma, which states that the absolute value of a first eigenfunction is a first eigenfunction.

Lemma 2.14. *Let $u \in N_0^{1,p}(\Omega)$ be a first eigenfunction of the p-Laplacian in Ω. Then $|u|$ is a first eigenfunction of the p-Laplacian in Ω.*

Proof. It is obvious that $|u| \in N_0^{1,p}(\Omega)$. Note also that in Definition 2.9 we could replace the integral of the minimal upper gradient g_v with

$$\inf \int_\Omega g^p \, d\mu,$$

where the infimum is taken over all upper gradients of v. Let u be a first eigenfunction of the p-Laplacian, and let g be an upper gradient of u. Then

$$\Big| |u(x)| - |u(y)| \Big| \le |u(x) - u(y)| \le \int_\gamma g \, ds$$

for any $x, y \in \Omega$ and for any rectifiable path γ joining x and y in Ω. Therefore g is also an upper gradient of $|u|$. Since this holds for any upper gradient g of u, we have

$$\int_\Omega g_{|u|}^p \, d\mu \le \int_\Omega g_u^p \, d\mu.$$

Therefore $|u|$ minimizes the functional J. $\square$

General setup

From now on we assume that the complete metric measure space X is equipped with a doubling Borel regular measure for which the measure of every nonempty open set is positive and the measure of every bounded set is finite. Furthermore we assume that the space supports a weak $(1,p)$-Poincaré inequality.

3 Boundedness

Let us first show that first eigenfunctions are bounded whenever $\Omega \subset X$ is bounded. The proof uses the method in Ladyzhenskaya–Ural'tseva [17, Lemma 5.1, p.71], see also Lindqvist [18]. Throughout this section, we fix a radius $R > 0$ such that $0 < 2R < \operatorname{diam}(X)$ and $\Omega \subset B_R \subset X$.

Theorem 3.1. *Let u be a first eigenfunction of the p-Laplacian in a bounded open set $\Omega \subset X$. Then u is bounded and satisfies the inequality*

$$\operatorname*{ess\,sup}_\Omega |u| \le c\lambda^{\kappa/\kappa-1} \int_\Omega |u| \, d\mu.$$

The constant c depends only on p, κ, c_μ, the measure of the ball B_R and the radius R.

Proof. Since $|u|$ is a first eigenfunction as well, we are free to assume that $u \ge 0$. We may also assume that u is not identically zero in Ω. Set

$$A_k = \{x \in \Omega \ : \ u(x) > k\}, \quad k \ge 0,$$

and denote

$$v = u - \max\{u - k, 0\} = u - (u - k)_+.$$

Since $u \in N_0^{1,p}(\Omega)$ is a first eigenfunction the inequality

$$\int_\Omega (g_u^p - \lambda |u|^p)\, d\mu \leq \int_\Omega (g_w^p - \lambda |w|^p)\, d\mu \tag{3.2}$$

holds for all $w \in N_0^{1,p}(\Omega)$. If we plug in the chosen v and (3.2), we have

$$\int_{A_k} g_u^p \, d\mu \leq \lambda \int_{A_k} (u^p - k^p)\, d\mu. \tag{3.3}$$

By the elementary inequality we have that $u^p \leq k^p + p(u-k)u^{p-1}$. Hence

$$\begin{aligned} \int_{A_k} g_u^p \, d\mu &\leq p\lambda \int_{A_k} (u-k)u^{p-1}\, d\mu \\ &\leq p2^{p-1}\lambda \int_{A_k} (u-k)^p\, d\mu + p2^{p-1}k^{p-1}\lambda \int_{A_k} (u-k)\, d\mu, \end{aligned} \tag{3.4}$$

where we used $(u-k)u^{p-1} \leq 2^{p-1}(u-k)^p + 2^{p-1}k^{p-1}(u-k)$ to obtain the second inequality. Since $(u-k)_+ \in N_0^{1,p}(B(z,R))$, Sobolev inequality (2.8) can be restated as

$$\Big(\int_{B_R} (u-k)_+^{\kappa p}\, d\mu \Big)^{1/\kappa p} \leq c \Big(\int_{B_R} g_{(u-k)_+}^p \, d\mu \Big)^{1/p},$$

where $c = c(p, \kappa, c_\mu, R, \mu(B_R)) > 0$. It follows that

$$\Big(\int_{A_k} (u-k)^{\kappa p}\, d\mu \Big)^{1/\kappa p} \leq c \Big(\int_{A_k} g_u^p \, d\mu \Big)^{1/p} \tag{3.5}$$

with the same constant c as above. Inequality (3.5) yields

$$\int_{A_k} (u-k)^p\, d\mu \leq c\mu(A_k)^{(\kappa-1)/\kappa} \int_{A_k} g_u^p\, d\mu, \tag{3.6}$$

where $c = c(p, \kappa, c_\mu, R, \mu(B_R)) > 0$. If we plug in (3.6) and (3.4) we have

$$\begin{aligned} \big(1 - c2^{p-1}\lambda\mu(A_k)^{(\kappa-1)/\kappa}\big) \int_{A_k} (u-k)^p\, d\mu \quad &\leq \quad c(2k)^{p-1}\lambda \\ &\quad \cdot \mu(A_k)^{(\kappa-1)/\kappa} \int_{A_k} (u-k)\, d\mu. \end{aligned}$$

Clearly we have that $k\mu(A_k) \leq \|u\|_{L^1(\Omega)}$. Therefore in the first factor on the left-hand side

$$c2^{p-1}\lambda\mu(A_k)^{(\kappa-1)/\kappa} \leq \frac{1}{2},$$

when $k \geq k_0 = (c2^p\lambda)^{\kappa/\kappa-1}\|u\|_{L^1(\Omega)}$. Using this and Hölder's inequality we obtain for $k \geq k_0$

$$\mu(A_k)^{1-p}\Big(\int_{A_k} (u-k)\, d\mu \Big)^p \leq c2^p k^{p-1}\lambda\mu(A_k)^{(\kappa-1)/\kappa} \int_{A_k} (u-k)\, d\mu,$$

from which we finally obtain

$$\int_{A_k} (u-k)\ d\mu \le (c2^p\lambda)^{1/(p-1)} k\mu(A_k)^{(\kappa p-1)/\kappa(p-1)} \tag{3.7}$$

when $k \ge k_0$. We need inequality (3.7) to bound u, see [17, Lemma 5.1, p.71]. The rest of the proof resembles somehow De Giorgi's argument. Writing

$$f(k) = \int_{A_k} (u-k)\ d\mu = \int_\Omega (u-k)_+\ d\mu = \int_k^\infty \mu(A_t)\ dt,$$

we have $f'(k) = -\mu(A_k)$ μ-almost everywhere and hence (3.7) can be written as

$$f(k) \le (c2^p\lambda)^{1/(p-1)} k(-f'(k))^{(\kappa p-1)/\kappa(p-1)}$$

μ-almost everywhere when $k \ge k_0$. If f is positive on the interval $[k_0, k]$ and we integrate the differential inequality from k_0 to k, we obtain

$$k^\alpha - k_0^\alpha \le (c2^p\lambda)^{\kappa/(\kappa p-1)} \left(f(k_0)^\alpha - f(k)^\alpha\right),$$

where $\alpha = (\kappa-1)/(\kappa p-1)$. This bounds k, since $0 \le f(k) \le f(k_0) \le f(0) = \|u\|_{L^1(\Omega)}$ on the right-hand side. Therefore, $f(k)$ is zero sooner or later. The quantitative bound is

$$k \le c2^{(2\kappa p-1)/(\kappa-1)} \lambda^{\kappa/(\kappa-1)} \int_\Omega |u|\ d\mu,$$

where $c = c(p, \kappa, c_\mu, R, \mu(B_R)) > 0$. This means that $f(k)$ is zero outside the given bound which implies

$$u \le c\lambda^{\kappa/(\kappa-1)} \int_\Omega |u|\ d\mu \tag{3.8}$$

μ-almost everywhere on Ω. Taking the essential supremum in (3.8) we get the desired result. If we consider the function $-u$, we get the bound for $\operatorname{ess\,inf} u$. □

4 Caccioppoli estimate and weak Harnack inequalities

In this section we prove weak Harnack's inequalities for non-negative first eigenfunctions. For this purpose we first establish a Caccioppoli type estimate and then apply Moser's iteration technique to obtain the upper weak Harnack inequality. The lower weak Harnack estimate follows from the results of Marola [19] together with the fact that non-negative first eigenfunctions are p-superminimizers of the p-energy integral, see Definition 2.9

Throughout this section we assume that $\Omega \subset X$ is bounded and we denote by R the radius of the ball B_R for which $\Omega \subset B_R \subset X$ and $0 < 2R < \operatorname{diam}(X)$.

Let us start by proving a suitable Caccioppoli type inequality.

Lemma 4.1. *Suppose that u is a non-negative first eigenfunction of the p-Laplacian in Ω and let $\varepsilon \geq 1$. Let η be a compactly supported Lipschitz continuous function in Ω such that $0 \leq \eta \leq 1$ and $g_\eta \leq C/r$. Then*

$$\int_{\operatorname{supp}(\eta)} g_u^p u^{\varepsilon-1} \eta^p \, d\mu \leq \frac{c}{r^p} \int_{\operatorname{supp}(\eta)} u^{p+\varepsilon-1} \, d\mu, \tag{4.2}$$

where $c = ((p/\varepsilon)^p C^p + (p/\varepsilon)\lambda R^p) > 0$.

Proof. Let η be a Lipschitz continuous function in Ω such that $\operatorname{supp}(\eta) \Subset \Omega$ and $0 \leq \eta \leq 1$. Since u is bounded due to Theorem 3.1, there is $0 < \alpha < \infty$ so that $\varepsilon \alpha^\varepsilon u^{\varepsilon-1} \leq 1$. Choosing

$$w = u - \eta^p (\alpha u)^\varepsilon,$$

we have $w \leq u$. Let Γ_{rect} denote the family of all rectifiable paths $\gamma : [0,1] \to X$. Let the family $\Gamma \subset \Gamma_{\text{rect}}$ be such that $\operatorname{Mod_p}(\Gamma) = 0$ and γ be the arc-length parametrization of the path in $\Gamma_{\text{rect}} \setminus \Gamma$ on which the function u is absolutely continuous. Since η is Lipschitz continuous, it is absolutely continuous on γ. We define $h : [0, l(\gamma)] \to [0, \infty)$,

$$h(s) = (u \circ \gamma)(s) - (\eta \circ \gamma)(s)^p (\alpha u \circ \gamma)(s)^\varepsilon.$$

Then h is absolutely continuous and for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$ we have

$$\begin{aligned} h'(s) &= (u \circ \gamma)'(s) - p(\eta \circ \gamma)(s)^{p-1} (\eta \circ \gamma)'(s) (\alpha u \circ \gamma)(s)^\varepsilon \\ &\quad - \varepsilon (\eta \circ \gamma)(s)^p (\alpha u \circ \gamma)(s)^{\varepsilon-1} \alpha (u \circ \gamma)'(s) \\ &= \left(1 - \varepsilon \alpha (\eta \circ \gamma)(s)^p (\alpha u \circ \gamma)(s)^{\varepsilon-1}\right) (u \circ \gamma)'(s) \\ &\quad - p(\eta \circ \gamma)(s)^{p-1} (\eta \circ \gamma)'(s) (\alpha u \circ \gamma)(s)^\varepsilon. \end{aligned}$$

Since $|(u \circ \gamma)'(s)| \leq g_u(\gamma(s))$ and $|(\eta \circ \gamma)'(s)| \leq g_\eta(\gamma(s))$ for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$, we obtain

$$\begin{aligned} |(w \circ \gamma)'(s)| = |h'(s)| &\leq \left(1 - \varepsilon \alpha \eta(\gamma(s))^p (\alpha u(\gamma(s)))^{\varepsilon-1}\right) g_u(\gamma(s)) \\ &\quad + p \eta(\gamma(s))^{p-1} (\alpha u(\gamma(s)))^\varepsilon g_\eta(\gamma(s)) \end{aligned}$$

for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$. Thus we have

$$g_w \leq \left(1 - \varepsilon \alpha^\varepsilon \eta^p u^{\varepsilon-1}\right) g_u + p \eta^{p-1} (\alpha u)^\varepsilon g_\eta$$

μ-almost everywhere in Ω. Since $0 \leq \varepsilon \alpha^\varepsilon \eta^p u^{\varepsilon-1} \leq 1$, we may exploit the convexity of the function $t \mapsto t^p$. We obtain

$$g_w^p \leq \left(1 - \varepsilon \alpha^\varepsilon \eta^p u^{\varepsilon-1}\right) g_u^p + \varepsilon^{1-p} p^p \alpha^\varepsilon u^{p+\varepsilon-1} g_\eta^p.$$

By the minimizing property of u, we have

$$\begin{aligned} \int_\Omega g_u^p \, d\mu &\leq \int_\Omega g_w^p \, d\mu + \lambda \int_\Omega (u^p - w^p) \, d\mu \\ &\leq \int_\Omega g_u^p \, d\mu - \varepsilon \alpha^\varepsilon \int_\Omega \eta^p u^{\varepsilon-1} g_u^p \, d\mu \\ &\quad + \varepsilon^{1-p} \alpha^\varepsilon p^p \int_\Omega u^{p+\varepsilon-1} g_\eta^p \, d\mu + \lambda \int_\Omega (u^p - w^p) \, d\mu, \end{aligned}$$

which implies

$$\int_{\mathrm{supp}(\eta)} \eta^p u^{\varepsilon-1} g_u^p \, d\mu \le \left(\frac{p}{\varepsilon}\right)^p \int_{\mathrm{supp}(\eta)} u^{p+\varepsilon-1} g_\eta^p \, d\mu + \frac{\lambda}{\varepsilon\alpha^\varepsilon} \int_\Omega (u^p - w^p) \, d\mu. \tag{4.3}$$

If we consider the last term on the right-hand side in more detail, we may write $u = w + \eta^p(\alpha u)^\varepsilon$ and use the elementary inequality $(a+b)^p \le a^p + pb(a+b)^{p-1}$ to obtain

$$\begin{aligned} \int_\Omega (u^p - w^p) \, d\mu &= \int_\Omega \left((w + \eta^p(\alpha u)^\varepsilon)^p - w^p\right) \, d\mu \\ &\le \int_\Omega \left(w^p + p\eta^p(\alpha u)^\varepsilon u^{p-1} - w^p\right) \, d\mu \\ &= \int_{\mathrm{supp}(\eta)} p\eta^p \alpha^\varepsilon u^{p+\varepsilon-1} \, d\mu. \end{aligned} \tag{4.4}$$

If we plug in (4.4) and (4.3) and use the fact that $g_\eta \le C/r$ we have

$$\int_{\mathrm{supp}(\eta)} \eta^p u^{\varepsilon-1} g_u^p \, d\mu \le \left(\left(\frac{p}{\varepsilon}\right)^p C^p + \frac{p}{\varepsilon}\lambda R^p\right) \frac{1}{r^p} \int_{\mathrm{supp}(\eta)} u^{p+\varepsilon-1} \, d\mu,$$

which is the desired estimate. □

Remark 4.5. The estimate of Lemma 4.1 actually holds also for $0 < \varepsilon < 1$. In fact, the proof above works in verbatim once we have shown that the function u is stricly positive and continuous, see Theorem 5.1 and Corollary 5.6 below. The point is that we may then choose a constant $\alpha > 0$ such that

$$0 \le \varepsilon\alpha^\varepsilon u^{\varepsilon-1} \le 1.$$

Moser's iteration argument yields the following weak Harnack inequality.

Theorem 4.6. *Suppose that u is a non-negative first eigenfunction of the p-Laplacian in Ω. Then for every ball $B(z,r)$ with $B(z,2r) \subset \Omega$ and any $q > 0$ we have*

$$\operatorname*{ess\,sup}_{B(z,r)} u \le c \left(\fint_{B(z,2r)} u^q \, d\mu \right)^{1/q}, \tag{4.7}$$

where $0 < c = c(p, q, \kappa, c_\mu, \lambda, R) < \infty$.

Proof. First we assume that $q \ge p$. Write $B_l = B(z, r_l), r_l = (1 + 2^{-l})r$ for $l = 0, 1, 2, \ldots$, thus, $B(z, 2r) = B_0 \supset B_1 \supset B_2 \supset \ldots$ Let η_l be a Lipschitz continuous function such that $0 \le \eta_l \le 1$, $\eta_l = 1$ on $\overline{B}_{l+1}$, $\eta_l = 0$ in $X \setminus B_l$ and $g_{\eta_l} \le 4 \cdot 2^l/r$ (choose, e.g., $\eta_l(x) = \min(\max((r_l - d(x,z))/(r_l - r_{l+1}))_+, 1))$. Fix $1 \le t < \infty$ and let

$$w_l = \eta_l u^{1+(t-1)/p} = \eta_l u^{\tau/p},$$

where $\tau := p + t - 1$. As in the proof of Lemma 4.1 for μ-almost everywhere in Ω we have

$$g_{w_l} \le g_{\eta_l} u^{\tau/p} + \frac{\tau}{p} u^{(t-1)/p} g_u \eta_l$$

and consequently

$$g_{w_l}^p \le 2^{p-1} g_{\eta_l}^p u^\tau + 2^{p-1}\left(\frac{\tau}{p}\right)^p u^{t-1} g_u^p \eta_l^p$$

μ-almost everywhere in Ω. By using the Caccioppoli estimate, Lemma 4.1, with $\varepsilon = t$ and $g_{\eta_l} \le 4 \cdot 2^l/r$ we obtain

$$\begin{aligned}
\left(\fint_{B_l} g_{w_l}^p \, d\mu\right)^{1/p} &\le 2^{(p-1)/p}\left(\fint_{B_l} \left(g_{\eta_l}^p u^\tau + \left(\frac{\tau}{p}\right)^p u^{t-1} g_u^p \eta_l^p\right) \, d\mu\right)^{1/p} \\
&\le 2(1+\lambda R^p)\tau \frac{4 \cdot 2^l}{r}\left(\fint_{B_l} u^\tau \, d\mu\right)^{1/p}.
\end{aligned}$$

The Sobolev inequality (2.8) implies

$$\begin{aligned}
\left(\fint_{B_l} w_l^{\kappa p} \, d\mu\right)^{1/\kappa p} &\le c(p, c_\mu) r_l \left(\fint_{B_l} g_{w_l}^p \, d\mu\right)^{1/p} \\
&\le c(p, c_\mu)(1+\lambda R^p)\tau(1+2^{-l}) r \frac{2^l}{r}\left(\fint_{B_l} u^\tau \, d\mu\right)^{1/p} \\
&\le c(p, c_\mu, \lambda, R)\tau 2^l \left(\fint_{B_l} u^\tau \, d\mu\right)^{1/p}
\end{aligned}$$

Since $w_l = u^{\tau/p}$ on B_{l+1}, by the doubling property of μ we obtain

$$\left(\fint_{B_{l+1}} u^{\kappa\tau} \, d\mu\right)^{1/\kappa\tau} \le \left(c(p, c_\mu, \lambda, R)\tau 2^l\right)^{p/\tau}\left(\fint_{B_l} u^\tau \, d\mu\right)^{1/\tau}.$$

This estimate holds for all $\tau \ge p$. We apply the estimate with $\tau = q\kappa^l$ for all $l = 0, 1, 2, \ldots$, we have

$$\left(\fint_{B_{l+1}} u^{q\kappa^{l+1}} \, d\mu\right)^{1/q\kappa^{l+1}} \le \left(c(p, c_\mu, \lambda, R)(q\kappa^l)2^l\right)^{p/q\kappa^l}\left(\fint_{B_l} u^{q\kappa^l} \, d\mu\right)^{1/q\kappa^l}.$$

By iterating we obtain the desired estimate

$$\begin{aligned}
\operatorname*{ess\,sup}_{B(z,r)} u &\le \left(c(p, c_\mu, \lambda, R)^{\sum_{i=0}^\infty \kappa^{-i}} \prod_{i=0}^\infty 2^{i\kappa^{-i}} \prod_{i=0}^\infty (q\kappa^i)^{\kappa^{-i}}\right)^{p/q}\left(\fint_{B(z,2r)} u^q \, d\mu\right)^{1/q} \\
&\le \left((c(p, c_\mu, \lambda, R)q)^{\kappa/(\kappa-1)}(2\kappa)^{\kappa/(\kappa-1)^2}\right)^{p/q}\left(\fint_{B(z,2r)} u^q \, d\mu\right)^{1/q} \\
&\le c(p, q, \kappa, c_\mu, \lambda, R)\left(\fint_{B(z,2r)} u^q \, d\mu\right)^{1/q}. \qquad (4.8)
\end{aligned}$$

The theorem is proved for $q \ge p$.

By the doubling property of the measure and (2.5), it is easy to see that (4.8) can be reformulated in a bit different manner. Namely, if $0 \le \rho < \widetilde{r} \le 2r$, then

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \le \frac{c}{(1-\rho/\widetilde{r})^{Q/q}} \left(\fint_{B(z,\widetilde{r})} u^q \, d\mu \right)^{1/q}, \tag{4.9}$$

where $0 < c = c(p, q, \kappa, c_\mu, \lambda, R) < \infty$. See Remark 4.4 in Kinnunen–Shanmugalingam [13].

If $0 < q < p$ we want to prove that there is a postive constant c so that

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \le \frac{c}{(1-\rho/2r)^{Q/q}} \left(\fint_{B(z,2r)} u^q \, d\mu \right)^{1/q},$$

when $0 \le \rho < 2r < \infty$. Now suppose that $0 < q < p$ and let $0 \le \rho < \widetilde{r} \le 2r$. We choose $q = p$ in (4.9), then

$$\begin{aligned}\operatorname*{ess\,sup}_{B(z,\rho)} u &\le \frac{c}{(1-\rho/\widetilde{r})^{Q/p}} \left(\fint_{B(z,\widetilde{r})} u^q u^{p-q} \, d\mu \right)^{1/p} \\ &\le \frac{c}{(1-\rho/\widetilde{r})^{Q/p}} \big(\operatorname*{ess\,sup}_{B(z,\widetilde{r})} u \big)^{1-q/p} \left(\fint_{B(z,\widetilde{r})} u^q \, d\mu \right)^{1/p}\end{aligned}$$

By Young's inequality

$$\begin{aligned}\operatorname*{ess\,sup}_{B(z,\rho)} u &\le \frac{p-q}{p} \operatorname*{ess\,sup}_{B(z,\widetilde{r})} u + \frac{c}{(1-\rho/\widetilde{r})^{Q/q}} \left(\fint_{B(z,\widetilde{r})} u^q \, d\mu \right)^{1/q} \\ &\le \frac{p-q}{p} \operatorname*{ess\,sup}_{B(z,\widetilde{r})} u + \frac{c}{(\widetilde{r}-\rho)^{Q/q}} \left((2r)^Q \fint_{B(z,2r)} u^q \, d\mu \right)^{1/q},\end{aligned}$$

where the doubling property (2.5) was used to obtain the last inequality. We need to get rid of the first term on the right-hand side. By applying a technical lemma, see Giaquinta [4, Lemma 3.1, p. 161], we obtain

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \le \frac{c}{(1-\rho/2r)^{Q/q}} \left(\fint_{B(z,2r)} u^q \, d\mu \right)^{1/q}$$

for all $0 \le \rho < 2r$, where $0 < c = c(p, q, \kappa, c_\mu, \lambda, R) < \infty$. If we set $\rho = r$, we obtain (4.8) for every $0 < q < p$ and the proof is complete. □

Remark 4.10. The statement of Theorem 4.6 was originally proved in Marola [19] for minimizers of the p-energy integral. However, the subminimizing property is not really needed. As our proof shows, it is enough to have a Caccioppoli type estimate in the spirit of (4.2).

The following lemma states that non-negative first eigenfunctions are p-superminimizers in the sense of Definition 2.9.

Lemma 4.11. *Let u be a non-negative first eigenfunction of the p-Laplacian in Ω. Then u is a p-superminimizer in Ω.*

Proof. Let $\Omega' \Subset \Omega$ be open and let $v \in N^{1,p}(\Omega)$ such that $v - u \in N_0^{1,p}(\Omega')$ and $v \geq u$ μ-almost everywhere in Ω'. Define

$$\psi = \begin{cases} v, & \mu\text{-a.e. in } \Omega' \\ u, & \mu\text{-a.e. in } \Omega \setminus \Omega' \end{cases}$$

Since $u \in N_0^{1,p}(\Omega)$, we have $\psi \in N_0^{1,p}(\Omega)$. Moreover,

$$\int_\Omega \psi^p \, d\mu \geq \int_\Omega u^p \, d\mu.$$

By the minimizing property of u

$$\begin{aligned} \int_{\Omega\setminus\Omega'} g_u^p \, d\mu + \int_{\Omega'} g_u^p \, d\mu &= \int_\Omega g_u^p \, d\mu \\ &\leq \left(\int_\Omega g_\psi^p \, d\mu \right) \left(\int_\Omega \psi^p \, d\mu \right)^{-1} \left(\int_\Omega u^p \, d\mu \right) \leq \int_\Omega g_\psi^p \, d\mu \\ &= \int_{\Omega\setminus\Omega'} g_u^p \, d\mu + \int_{\Omega'} g_v^p \, d\mu. \end{aligned}$$

Hence

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu,$$

and we are done. □

Lemma 4.11 yields together with results of [19] the following weak Harnack inequality.

Theorem 4.12. *Let u be a non-negative first eigenfunction of the p-Laplacian in Ω. Then there are $q > 0$ and $c = c(p, q, \kappa, c_\mu) > 0$ such that*

$$\left(\fint_{B(z,2r)} u^q \, d\mu \right)^{1/q} \leq c \operatorname*{ess\,inf}_{B(z,r)} u \tag{4.13}$$

for every ball $B(z, r)$ such that $B(z, 10\tau r) \subset \Omega$.

Proof. By Lemma 4.11, u is a p-superminimizer in Ω. It is evident that $u + \beta$ is a p-superminimizer in Ω for all constants $\beta > 0$. Hence we may apply [19, Theorem 5.19] to obtain that

$$\left(\fint_{B(z,2r)} (u + \beta)^q \, d\mu \right)^{1/q} \leq c \operatorname*{ess\,inf}_{B(z,r)} (u + \beta)$$

for all $\beta > 0$ and for every ball $B(z, r)$ such that $B(z, 10\tau r) \subset \Omega$. The claim follows by letting $\beta \to 0+$. □

The constant $\tau \geq 1$ comes from the weak $(1, p)$-Poincaré inequality (2.6).

5 Continuity and Harnack's inequality

We first give a simple proof for the continuity of u by combining the upper weak Harnack estimate of the De Giorgi method together with the lower weak Harnack estimate (4.13). Observe here that only $(1, p)$-Poincare inequality is needed for the estimate in Pere [22, Theorem 5.18, p. 15]. Next, for a function u we let

$$\operatorname{ess}\liminf_{x\to z} u(x) = \lim_{r\to 0} \operatorname{ess}\inf_{B(z,r)} u.$$

Theorem 5.1. *The first eigenfunction u is continuous in Ω.*

Proof. By Lemma 2.14, we are free to assume that u is non-negative. Let $z \in \Omega$ and let $m_r = \operatorname{ess}\inf_{B(z,r)} u$ for sufficiently small radii r. The same argumentation as in Heinonen–Kilpeläinen–Martio [9, pp. 76–77] yields that

$$\lim_{r\to 0} \fint_{B(z,r)} (u - m_r)\, d\mu = 0 \tag{5.2}$$

and that

$$\operatorname{ess}\liminf_{x\to z} u(x) = \lim_{r\to 0} \fint_{B(z,r)} u\, d\mu. \tag{5.3}$$

Define u pointwise by (5.3). Then u is lower semicontinuous, $u - m_r$ is non-negative in B_r, and

$$\fint_{B(z,r)} |u - u(z)|\, d\mu = \fint_{B(z,r)} |u - m_r|\, d\mu + \fint_{B(z,r)} |m_r - u(z)|\, d\mu.$$

By (5.2) and (5.3), both terms on the right hand side tend to zero as $r \to 0$. Hence we conclude that u has Lebesgue points everywhere in Ω. Since u is bounded, we get

$$\lim_{r\to 0} \fint_{B(z,r)} |u - u(z)|^p\, d\mu = 0 \tag{5.4}$$

by using the trivial estimate

$$\fint_{B(z,r)} |u - u(z)|^p\, d\mu \le \Big(\sup_{B(z,r)} u\Big)^{p-1} \fint_{B(z,r)} |u - u(z)|\, d\mu$$

for small radii. Next, we recall the estimate

$$\operatorname{ess}\sup_{B(z,r/2)} u \le k_0(1+r) + c\bigg(\fint_{B(z,r)} ((u-k_0)^+)^p\, d\mu\bigg)^{1/p} \left(\frac{\mu(A(k_0,r))}{\mu(B(z,r)}\right)^{\alpha/p}$$

from [22, p. 15]. Here k_0 is any non-negative number and $A(k_0, r) = B(z, r) \cap \{\, u > k_0 \}$. Now it is enough to choose $k_0 = u(z)$, let $r \to 0$, and use (5.4) to conclude that

$$\operatorname{ess}\limsup_{x\to z} u(x) \le u(z).$$

Hence u is also upper semicontinuous and the claim follows. $\square$

Combining Theorem 4.6 and Theorem 4.12 we obtain immediately Harnack's inequality.

Theorem 5.5. *Let u be a non-negative first eigenfunction of the p-Laplacian in Ω. Then there exists a constant $c = c(p, q, \kappa, c_\mu, \lambda, R) > 0$ so that*

$$\sup_{B(z,r)} u \le c \inf_{B(z,r)} u$$

for every ball $B(z, r)$ for which $B(z, 10\tau r) \subset \Omega$. The constant c is independent of the ball $B(z, r)$ and the function u. The constant $\tau \ge 1$ comes from the weak $(1, p)$-Poincaré inequality.

Observe here that continuity does not follow easily from Harnack's inequality since the sum of an eigenfunction and a constant is not an eigenfunction in general.

By continuity and Harnack's inequality we obtain that first eigenfunctions do not change sign in any bounded domain.

Corollary 5.6. *Let u be a non-negative first eigenfunction in a bounded domain $\Omega \subset X$. Then u is strictly positive in Ω.*

Proof. Denote $U_1 = \{ z \in \Omega : u(z) > 0 \}$, $U_2 = \{ z \in \Omega : u(z) = 0 \}$, and assume that U_1 and U_2 are both non-empty. By connectedness, we are free to assume that at least one of the sets U_1 and U_2 is not open. If U_1 is not open, there is $z \in U_1$ which does not belong to the interior of U_1. Hence, for some $r > 0$, we may apply the Harnack's inequality to conclude that

$$\sup_{B(z,r)} u \le c \inf_{B(z,r)} u = 0.$$

This contradicts the fact that $x \in U_1$. The case that U_2 is not open is treated similarly. $\square$

Corollary 5.7. *Let u be a first eigenfunction in a bounded domain $\Omega \subset X$. Then u does not change sign in Ω.*

Proof. Since $|u|$ is a non-negative first eigenfunction in Ω, Corollary 5.6 implies that

$$\Omega = \{ u > 0 \} \cap \{ u < 0 \}.$$

By continuity, both $\{ u > 0 \}$ and $\{ u < 0 \}$ is open. The assumption that both level sets are non-empty contradicts the connectedness. $\square$

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