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Abstract: *We consider the Mixed Interpolated (Tensorial Components) finite element families for the Reissner–Mindlin plate model. For the case of a convex domain with clamped boundary conditions we prove regularity results and derive new error estimates which are uniformly valid with respect to the thickness parameter.*

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1 Introduction

One of the most successful finite element methods for the Reissner–Mindlin plate bending model is the MITC family introduced by K.-J. Bathe and coworkers [8], [6], [7]. The family has been mathematically analyzed under various assumptions: The first error analysis [7] was performed for the limiting case of a vanishing thickness and in [10], [12] the analysis was extended to a positive thickness. The result is, roughly speaking, that the error is quasi-optimal in the sense that the finite element error is bounded from above by a constant times the interpolation error, and it is essential that the constant is independent of the plate thickness. The estimate is, however, somewhat unsatisfactory, because it is combined with an interpolation estimate obtained by assuming a smooth solution. In practice, the solution is never very smooth since it is known [1], [3], [4] that the solution contains strong boundary layers. In [13] an analysis is performed by taking the boundary layer into account for the free plate with a smooth boundary.

The purpose of this paper is to make an analysis in spirit of [13], but now for a clamped plate and a polygonal domain. We prove an estimate uniformly valid and, in particular, we give the estimate with respect to the loading. For this we prove a regularity result that can be used for the analysis of other finite elements as well. For simplicity, we consider a triangular MITC method, but it is clear that the results also hold for other elements of the MITC-type, such as the triangular and quadrilateral families reviewed in [10], [15]. The results of this paper are used in [11] where a postprocessing method is introduced, analyzed and tested.

2 The Reissner–Mindlin plate model

We consider a clamped plate with the midsurface $\Omega \subset \mathbb{R}^2$ and scale the loading f by assuming it to be of the form $f = Gt^3 g$, with G denoting the shear modulus and t denoting the thickness. This gives a well posed problem in the limit $t \to 0$, cf. [9].

We define the bilinear form

$$\mathcal{B}(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = a(\boldsymbol{\phi}, \boldsymbol{\eta}) + t^{-2}(\nabla z - \boldsymbol{\phi}, \nabla v - \boldsymbol{\eta}), \tag{2.1}$$

with

$$a(\boldsymbol{\phi}, \boldsymbol{\eta}) = \frac{1}{6}\{(\boldsymbol{\varepsilon}(\boldsymbol{\phi}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})) + \frac{\nu}{1-\nu}(\operatorname{div} \boldsymbol{\phi}, \operatorname{div} \boldsymbol{\eta})\}, \tag{2.2}$$

where ν is the Poisson ratio and $\boldsymbol{\varepsilon}$ is the linear strain tensor

$$\boldsymbol{\varepsilon}(\boldsymbol{\eta}) = \frac{1}{2}(\nabla \boldsymbol{\eta} + (\nabla \boldsymbol{\eta})^T). \tag{2.3}$$

The problem is then the following:

Variational formulation 2.1. *Find the deflection $w \in H_0^1(\Omega)$ and the rotation $\boldsymbol{\beta} \in [H_0^1(\Omega)]^2$ such that*

$$\mathcal{B}(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = (g, v) \ \ \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2. \tag{2.4}$$

For the analysis the problem is first written in a mixed form with the shear force

$$\boldsymbol{q} = \frac{1}{t^2}(\nabla w - \boldsymbol{\beta}) \tag{2.5}$$

taken as an independent unknown in the space $[L^2(\Omega)]^2$ (cf. [10]). This gives the following problem:

Variational formulation 2.2. *Find* $(w, \boldsymbol{\beta}, \boldsymbol{q}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^2$ *such that*

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) = (g, v) \ \ \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2, \tag{2.6}$$

$$(\nabla w - \boldsymbol{\beta}, \boldsymbol{r}) - t^2(\boldsymbol{q}, \boldsymbol{r}) = 0 \ \ \forall \boldsymbol{r} \in [L^2(\Omega)]^2. \tag{2.7}$$

For further reference, we give our regularity results with a more general right hand side:

Variational formulation 2.3. *Find* $(w, \boldsymbol{\beta}, \boldsymbol{q}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^2$ *such that*

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) = (g, v) + (\boldsymbol{G}, \boldsymbol{\eta}) \tag{2.8}$$

for all $(v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2$*, and*

$$(\nabla w - \boldsymbol{\beta}, \boldsymbol{r}) - t^2(\boldsymbol{q}, \boldsymbol{r}) = 0 \ \ \forall \boldsymbol{r} \in [L^2(\Omega)]^2. \tag{2.9}$$

For clarifying the detailed regularity structure of the problem the Helmholtz decomposition is used for the shear force. In [9] it is proved that for $\boldsymbol{q} \in [L^2(\Omega)]^2$ one can write

$$\boldsymbol{q} = \nabla \psi + \mathbf{rot}\, p, \tag{2.10}$$

with a unique pair $(\psi, p) \in H_0^1(\Omega) \times [H^1(\Omega) \cap L_0^2(\Omega)]$, and the following orthogonality holds

$$(\nabla \psi, \mathbf{rot}\, p) = 0. \tag{2.11}$$

By using the same orthogonal splitting for the test function

$$\boldsymbol{r} = \nabla v + \mathbf{rot}\, q \tag{2.12}$$

the above formulations are equivalent to the following problem:

Variational formulation 2.4. *Find* $(w, \boldsymbol{\beta}, \psi, p) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \times H_0^1(\Omega) \times [H^1(\Omega) \cap L_0^2(\Omega)]$ *such that*

$$(\nabla \psi, \nabla \varphi) = (g, \varphi) \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.13}$$

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) - (\mathbf{rot}\, p, \boldsymbol{\eta}) = (\nabla \psi, \boldsymbol{\eta}) + (\boldsymbol{G}, \boldsymbol{\eta}) \qquad \forall \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \tag{2.14}$$

$$t^2(\mathbf{rot}\, p, \mathbf{rot}\, q) + (\boldsymbol{\beta}, \mathbf{rot}\, q) = 0 \qquad \forall q \in H^1(\Omega) \cap L_0^2(\Omega), \tag{2.15}$$

$$(\nabla w, \nabla v) = (\boldsymbol{\beta}, \nabla v) + t^2(\nabla \psi, \nabla v) \ \ \forall v \in H_0^1(\Omega). \tag{2.16}$$

In the limit $t \to 0$ the solution $(w, \boldsymbol{\beta}) = (w^t, \boldsymbol{\beta}^t)$ converges to the Kirchhoff limit with

$$\boldsymbol{\beta}^0 = \nabla w^0. \tag{2.17}$$

We write

$$w = w^0 + w^r \ \text{ and } \ \boldsymbol{\beta} = \boldsymbol{\beta}^0 + \boldsymbol{\beta}^r. \tag{2.18}$$

We now prove the following global and interior regularity estimates:

Theorem 2.1. *Let Ω be a convex polygonal domain and let Ω_i be a domain compactly embedded in Ω. With $g \in H^{s-2}(\Omega)$ and $tg \in H^{s-1}(\Omega)$, $s \geq 1$, it then holds*

$$\begin{aligned}&\|w^0\|_3 + t^{-1}\|w^r\|_2 + \|\boldsymbol{\beta}\|_2 + \|\psi\|_1 + \|p\|_1 + t\|p\|_2 \\ &\quad \leq C(\|g\|_{-1} + t\|g\|_0 + \|\boldsymbol{G}\|_0)\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\|w^0\|_{s+2,\Omega_i} + t^{-1}\|w^r\|_{s+1,\Omega_i} + \|\boldsymbol{\beta}\|_{s+1,\Omega_i} + \|\psi\|_{s,\Omega_i} + \|p\|_{s,\Omega_i} + t\|p\|_{s+1,\Omega_i} \\ &\leq C(\|g\|_{s-2} + t\|g\|_{s-1} + \|\boldsymbol{G}\|_{s-1}).\end{aligned} \tag{2.20}$$

Proof. Step 1. As w^0 is the Kirchhoff solution it is clear that

$$\|w^0\|_3 \leq C\|g\|_{-1} \text{ and } \|w^0\|_{s+2,\Omega_i} \leq C\|g\|_{s-2}. \tag{2.21}$$

Let now $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, (with $\subset\subset$ denoting a compact embedding) be arbitrary. For the solution of the Poisson problem (2.13) above we have

$$\|\psi\|_l \leq C\|g\|_{l-2}, \text{ for } l = 1, 2, \text{ and } \|\psi\|_{s,\Omega''} \leq C\|g\|_{s-2}. \tag{2.22}$$

Step 2. To obtain the other estimates we rely on the results proved by Arnold, Falk and Liu [2], [5] for the following general problem: find $(\boldsymbol{\Phi}, P) \in [H^1_0(\Omega)]^2 \times [H^1(\Omega) \cap L^2_0(\Omega)]$ such that

$$a(\boldsymbol{\Phi}, \boldsymbol{\eta}) - (\mathbf{rot}\, P, \boldsymbol{\eta}) = (\boldsymbol{F}, \boldsymbol{\eta}), \qquad \boldsymbol{\eta} \in [H^1_0(\Omega)]^2, \tag{2.23}$$

$$t^2(\mathbf{rot}\, P, \mathbf{rot}\, q) + (\boldsymbol{\Phi}, \mathbf{rot}\, q) = (K, q), \qquad q \in H^1(\Omega) \cap L^2_0(\Omega). \tag{2.24}$$

For this problem the following estimates are in essence proved in [2], [5]:

$$\|\boldsymbol{\Phi}\|_2 + \|P\|_1 + t\|P\|_2 + t^2\|P\|_3 \leq C(\|\boldsymbol{F}\|_0 + \|K\|_1) \tag{2.25}$$

and

$$\begin{aligned}&\|\boldsymbol{\Phi}\|_{l+2,\Omega'} + \|P\|_{l+1,\Omega'} + t\|P\|_{l+2,\Omega'} + t^2\|P\|_{l+3,\Omega'} \\ &\leq C(\|\boldsymbol{F}\|_{l,\Omega''} + \|K\|_{l+1,\Omega''}),\end{aligned} \tag{2.26}$$

with $l \geq 0$. Applying these results with $\boldsymbol{\Phi} = \boldsymbol{\beta}$, $P = p$, $\boldsymbol{F} = \nabla\psi + \boldsymbol{G}$, $K = 0$, $\Omega' = \Omega_i$, and $l = s - 1$, gives

$$\|\boldsymbol{\beta}\|_2 + \|p\|_1 + t\|p\|_2 \leq C\big(\|\nabla\psi\|_0 + \|\boldsymbol{G}\|_0\big) \tag{2.27}$$

and

$$\|\boldsymbol{\beta}\|_{s+1,\Omega_i} + \|p\|_{s,\Omega_i} + t\|p\|_{s+1,\Omega_i} \leq C\big(\|\nabla\psi\|_{s-1,\Omega''} + \|\boldsymbol{G}\|_{s-1,\Omega''}\big). \tag{2.28}$$

Hence, we obtain all of the asserted estimates except the one for w^r. To this end, let $(\boldsymbol{\beta}^0, p^0)$ be the solution of (2.14)–(2.15) with $t = 0$. From the results in [2], [5] one obtains

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}^0\|_1 \leq Ct\big(\|\nabla\psi\|_0 + \|\boldsymbol{G}\|_0\big). \tag{2.29}$$

The difference $w^r = w - w^0$ satisfies

$$(\nabla w^r, \nabla v) = (\boldsymbol{\beta} - \boldsymbol{\beta}^0, \nabla v) + t^2(g, v), \quad v \in H_0^1(\Omega), \tag{2.30}$$

and hence the H^2-regularity for the Poisson problem gives

$$\|w^r\|_2 \leq C\big(\|\boldsymbol{\beta} - \boldsymbol{\beta}^0\|_1 + t^2\|g\|_0\big). \tag{2.31}$$

Combining this with (2.29) and (2.22) gives the remaining part of the global regularity estimate (2.19).

Step 3. Next, let us derive the remaining local estimates in (2.20). From (2.14)–(2.15) we obtain

$$a(\boldsymbol{\beta} - \boldsymbol{\beta}^0, \boldsymbol{\eta}) - (\mathbf{rot}\,(p - p^0), \boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \tag{2.32}$$

$$(\boldsymbol{\beta} - \boldsymbol{\beta}^0, \mathbf{rot}\, q) = -t^2(\mathbf{rot}\, p, \mathbf{rot}\, q), \qquad q \in H^1(\Omega) \cap L_0^2(\Omega). \tag{2.33}$$

From (2.15) it follows that p satisfies the natural boundary condition $\partial p/\partial n = 0$ on $\partial\Omega$, and the right hand side of (2.33) above is equal to $t^2(\Delta p, q)$. Hence, applying the estimate (2.26) with $\boldsymbol{\Phi} = \boldsymbol{\beta} - \boldsymbol{\beta}^0$, $P = p - p^0$, $\boldsymbol{F} = \mathbf{0}$, $K = t^2\Delta p$, $\Omega' = \Omega_i$ and $l = s - 2$, gives

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}^0\|_{s,\Omega_i} \leq Ct^2\|p\|_{s+1,\Omega''}. \tag{2.34}$$

Exactly in the same way as before we now get

$$t\|p\|_{s+1,\Omega''} \leq C\|g\|_{s-2} \tag{2.35}$$

and hence

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}^0\|_{s,\Omega_i} \leq Ct\|g\|_{s-2}. \tag{2.36}$$

The equation (2.30) and the interior elliptic regularity for the Poisson problem then gives

$$\begin{aligned} \|w^r\|_{s+1,\Omega_i} &\leq C\big(\|\boldsymbol{\beta} - \boldsymbol{\beta}^0\|_{s,\Omega_i} + t^2\|g\|_{s-1}\big) \\ &\leq Ct\big(\|g\|_{s-2} + t\|g\|_{s-1}\big), \end{aligned} \tag{2.37}$$

which concludes the proof. □

3 The MITC Finite Elements

Let us recall the triangular MITC elements [7], [10]. Let $\mathcal{C}_h$ be the triangulation of $\overline{\Omega}$ and let us denote $h = \max_{K \in \mathcal{C}_h} h_K$, where h_K is the diameter of K. $P_k(K)$ denotes the space of polynomials of degree k on K. Throughout the paper C denotes a positive constant independent of both the mesh size h and the plate thickness t.

The finite element subspaces $W_h \subset H_0^1(\Omega)$ and $\boldsymbol{V}_h \subset [H_0^1(\Omega)]^2$ are defined for the polynomial degree $k \geq 2$ as follows:

$$W_h = \{w \in H_0^1(\Omega) \ \mid \ w_{|K} \in P_k(K) \ \ \forall K \in \mathcal{C}_h\}, \tag{3.1}$$

$$\boldsymbol{V}_h = \{\boldsymbol{\eta} \in [H_0^1(\Omega)]^2 \ \mid \ \boldsymbol{\eta}_{|K} \in [P_k(K)]^2 \oplus [B_{k+1}(K)]^2 \ \ \forall K \in \mathcal{C}_h\}, \tag{3.2}$$

with the "bubble space"

$$B_{k+1}(K) = \{b = b_3 p \mid p \in \tilde{P}_{k-2}(K),\ b_3 \in P_3(K),\ b_{3|E} = 0\ \ \forall E \subset \partial K\}, \tag{3.3}$$

where $\widetilde{P}_{k-2}(K)$ is the space of homogenous polynomials of degree $k-2$ on the element K.

The rotated Raviart–Thomas space of order $k-1$ is denoted by

$$\boldsymbol{Q}_h = \{\, \boldsymbol{r} \in \boldsymbol{H}(\mathrm{rot};\Omega) \mid \boldsymbol{r}_{|K} \in [P_{k-1}(K)]^2 \oplus (y,-x)\tilde{P}_{k-1}(K)\ \forall K \in \mathcal{C}_h\,\}. \tag{3.4}$$

The reduction operator $\boldsymbol{R}_h : [H^1(\Omega)]^2 \to \boldsymbol{Q}_h$ is defined by the conditions

$$\langle (\boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta}) \cdot \boldsymbol{\tau}_E, p\rangle_E = 0\ \ \forall p \in P_{k-1}(E)\ \ \forall E \subset \partial K, \tag{3.5}$$

$$(\boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta}, \boldsymbol{p})_K = 0\ \ \forall \boldsymbol{p} \in [P_{k-2}(K)]^2. \tag{3.6}$$

Here E denotes an edge to K and $\boldsymbol{\tau}_E$ is the unit tangent to E. $(\cdot,\cdot)_K$ and $\langle\cdot,\cdot\rangle_E$ are the L^2-inner products.

The MITC method is now defined as follows:

Method 3.1. *Find the deflection $w_h \in W_h$ and the rotation $\boldsymbol{\beta}_h \in \boldsymbol{V}_h$ such that*

$$\mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = (g, v)\ \ \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{3.7}$$

with the modified bilinear form

$$\mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = a(\boldsymbol{\phi},\boldsymbol{\eta}) + \frac{1}{t^2}(\boldsymbol{R}_h(\nabla z - \boldsymbol{\phi}), \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta})). \tag{3.8}$$

The discrete shear force $\boldsymbol{q}_h \in \boldsymbol{Q}_h$ is

$$\boldsymbol{q}_h = \frac{1}{t^2}\boldsymbol{R}_h(\nabla w_h - \boldsymbol{\beta}_h). \tag{3.9}$$

Now, the mixed variant of Method 3.1 is of the following form [10]:

Method 3.2. *Find $(w_h, \boldsymbol{\beta}_h, \boldsymbol{q}_h) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{Q}_h \subset H^1_0(\Omega) \times [H^1_0(\Omega)]^2 \times [L^2(\Omega)]^2$ such that*

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + (\boldsymbol{q}_h, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta})) = (g, v)\ \ \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{3.10}$$

$$(\boldsymbol{R}_h(\nabla w_h - \boldsymbol{\beta}_h), \boldsymbol{r}) - t^2(\boldsymbol{q}_h, \boldsymbol{r}) = 0\ \ \forall \boldsymbol{r} \in \boldsymbol{Q}_h. \tag{3.11}$$

The key in the error analysis of the MITC elements performed in [10], [12] is that there exists a discrete Helmholtz decomposition. For this we define

$$P_h = \{\, q \in L^2_0(\Omega) \mid q_{|K} \in P_{k-1}(K)\ \forall K \in \mathcal{C}_h\,\}. \tag{3.12}$$

Lemma 3.1. *For every $\boldsymbol{r} \in \boldsymbol{Q}_h$ there exists unique $v \in W_h$, $q \in P_h$ and $\boldsymbol{\alpha} \in \boldsymbol{Q}_h$ such that*

$$\boldsymbol{r} = \nabla v + \boldsymbol{\alpha} \tag{3.13}$$

and

$$(\boldsymbol{\alpha}, \boldsymbol{s}) = (\mathrm{rot}\, \boldsymbol{s}, q) \quad \forall \boldsymbol{s} \in \boldsymbol{Q}_h. \tag{3.14}$$

The second relation motivates the notation

$$\boldsymbol{\alpha} = \mathbf{rot}_h\, q \tag{3.15}$$

and we have the orthogonality

$$(\mathbf{rot}_h\, q, \nabla v) = 0. \tag{3.16}$$

Note that this gives [12], with L^2-projections $\mathit{\Pi}_h : L_0^2(\Omega) \to P_h$ and $\boldsymbol{\mathit{\Pi}}_h : [L^2(\Omega)]^2 \to \boldsymbol{Q}_h$,

$$\mathbf{rot}_h \mathit{\Pi}_h q = \boldsymbol{\mathit{\Pi}}_h \mathbf{rot}\, q \qquad \forall q \in H^1(\Omega), \tag{3.17}$$

and for $q \in P_h$

$$(\mathbf{rot}_h\, q, \boldsymbol{R}_h \boldsymbol{\beta}_h) = (q, \mathrm{\mathbf{rot}}\, \boldsymbol{\beta}_h). \tag{3.18}$$

By using this result and writing

$$\boldsymbol{q}_h = \nabla \psi_h + \mathbf{rot}_h\, p_h \tag{3.19}$$

and

$$\boldsymbol{r} = \nabla v + \mathbf{rot}_h\, q \tag{3.20}$$

in Method 3.2 we get the equivalent formulation:

Method 3.3. *Find* $(w_h, \boldsymbol{\beta}_h, \psi_h, p_h) \in W_h \times \boldsymbol{V}_h \times W_h \times P_h$ *such that*

$$(\nabla \psi_h, \nabla \varphi) = (g, \varphi) \qquad \forall \varphi \in W_h, \tag{3.21}$$

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) - (p_h, \mathrm{rot}\, \boldsymbol{\eta}) = (\nabla \psi_h, \boldsymbol{R}_h \boldsymbol{\eta}) \quad \forall \boldsymbol{\eta} \in \boldsymbol{V}_h, \tag{3.22}$$

$$t^2(\mathbf{rot}_h\, p_h, \mathbf{rot}_h\, q) + (\mathrm{rot}\, \boldsymbol{\beta}_h, q) = 0 \qquad \forall q \in P_h, \tag{3.23}$$

$$(\nabla w_h, \nabla v) = (\boldsymbol{R}_h \boldsymbol{\beta}_h, \nabla v) + t^2(\nabla \psi_h, \nabla v) \quad \forall v \in W_h. \tag{3.24}$$

We now estimate the errors between the continuous Variational formulation 2.4 (with $\boldsymbol{G} = \mathbf{0}$) and the discrete Method 3.3. In W_h and $\boldsymbol{V}_h$ we use the standard Lagrange interpolants with the well-known estimates of the following type:

Lemma 3.2. *There is a positive constant* C *such that*

$$\|v - I_h v\|_{1,K} \le C h_K^{m-1} \|v\|_{m,K} \quad \forall v \in H^m(K), \tag{3.25}$$

where $2 \le m \le k+1$.

In the discrete shear space $\boldsymbol{Q}_h$ we use the Raviart–Thomas interpolation operator defined in (3.5)–(3.6) for which it holds:

Lemma 3.3. [14] *There is a positive constant* C *such that*

$$\|\boldsymbol{\eta} - \boldsymbol{R}_h \boldsymbol{\eta}\|_{0,K} \le C h_K^m \|\boldsymbol{\eta}\|_{m,K} \quad \forall \boldsymbol{\eta} \in [H^m(K)]^2, \tag{3.26}$$

where $1 \le m \le k$.

In order to have a measure of the influence of the boundary layer we use the following notation. With the interior region Ω_i we denote

$$\Omega_i^h = \cup_{K\subset\Omega_i} K, \qquad \Omega_b^h = \Omega \setminus \Omega_i^h \tag{3.27}$$

and

$$h_i = \max_{K\in\Omega_i^h} h_K, \qquad h_b = \max_{K\in\Omega_b^h} h_K. \tag{3.28}$$

Our error estimate is now the following:

Theorem 3.1. *Let Ω be a convex polygon and suppose that the plate is clamped. For $g \in H^{k-2}(\Omega)$, $tg \in H^{k-1}(\Omega)$ it then holds*

$$\begin{aligned} &\|w - w_h\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|\psi - \psi_h\|_1 + \|p - p_h\|_0 + t\|\mathbf{rot}\, p - \mathbf{rot}_h\, p_h\|_0 \\ &\quad \le C\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\} \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} &\|w - w_h\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_0 \\ &\quad \le Ch\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\}. \end{aligned} \tag{3.30}$$

Proof. Step 1. From (2.13) and (3.21) we get

$$\begin{aligned} |\psi - \psi_h|_1 &\le |\psi - I_h\psi|_1 \\ &\le |\psi - I_h\psi|_{1,\Omega_i^h} + |\psi - I_h\psi|_{1,\Omega_b^h}. \end{aligned} \tag{3.31}$$

In Ω_i^h we apply Lemma 3.2 with $m = k+1$ and in Ω_b^h with $m = 2$, and then we use Theorem 2.1 and obtain

$$\begin{aligned} &|\psi - I_h\psi|_{1,\Omega_i^h} + |\psi - I_h\psi|_{1,\Omega_b^h} \\ &\quad \le C\big(h_i^k\|\psi\|_{k+1,\Omega_i^h} + h_b\|\psi\|_{2,\Omega_b^h}\big) \\ &\quad \le C\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\}. \end{aligned} \tag{3.32}$$

Step 2. The equations (3.22)–(3.23) are a discretization of the singularly perturbed Stokes system (2.14)–(2.15) (with $\boldsymbol{G} = \boldsymbol{0}$). For this we have the stability in the norms $\|\cdot\|_1$ (for the rotation) and $\|\cdot\|_0 + t\|\mathbf{rot}_h\,(\cdot)\|_0$ (for the pressure). By the standard arguments, taking the non-consistency into account (cf. [12]), we get

$$\begin{aligned} &\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|p - p_h\|_0 + t\|\mathbf{rot}\, p - \mathbf{rot}_h\, p_h\|_0 \\ &\quad \le C\big(\|\boldsymbol{\beta} - \boldsymbol{I}_h\boldsymbol{\beta}\|_1 + \|p - \mathit{\Pi}_h p\|_0 + t\|\mathbf{rot}\, p - \mathbf{rot}_h\, \mathit{\Pi}_h p\|_0 \\ &\qquad + |\psi - \psi_h|_1 + \sup_{\boldsymbol{\eta}\in\boldsymbol{V}_h} \frac{|(\nabla\psi_h, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_1}\big). \end{aligned} \tag{3.33}$$

The first two terms above we estimate as before by using the interpolation estimates and Theorem 2.1

$$\begin{aligned}
&\|\boldsymbol{\beta} - \boldsymbol{I}_h\boldsymbol{\beta}\|_1 + \|p - \Pi_h p\|_0 \\
&\quad \le \|\boldsymbol{\beta} - \boldsymbol{I}_h\boldsymbol{\beta}\|_{1,\Omega_i^h} + \|p - \Pi_h p\|_{0,\Omega_i^h} \\
&\qquad + \|\boldsymbol{\beta} - \boldsymbol{I}_h\boldsymbol{\beta}\|_{1,\Omega_b^h} + \|p - \Pi_h p\|_{0,\Omega_b^h} \\
&\quad \le C h_i^k \big(\|\boldsymbol{\beta}\|_{k+1,\Omega_i^h} + \|p\|_{k,\Omega_i^h}\big) + C h_b \big(\|\boldsymbol{\beta}\|_{2,\Omega_b^h} + \|p\|_{1,\Omega_b^h}\big) \\
&\quad \le C\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\}.
\end{aligned} \tag{3.34}$$

For the third term the relation (3.17) and the interpolation estimate give

$$\begin{aligned}
&t\|\mathbf{rot}\, p - \mathbf{rot}_h\, \Pi_h p\|_0 = t\|\mathbf{rot}\, p - \boldsymbol{\Pi}_h \mathbf{rot}\, p\|_0 \\
&\quad \le t\|\mathbf{rot}\, p - \boldsymbol{\Pi}_h \mathbf{rot}\, p\|_{0,\Omega_i^h} + t\|\mathbf{rot}\, p - \boldsymbol{\Pi}_h \mathbf{rot}\, p\|_{0,\Omega_b^h} \\
&\quad \le C\big(h_i^k \|t\, \mathbf{rot}\, p\|_{k,\Omega_i^h} + h_b \|t\, \mathbf{rot}\, p\|_{1,\Omega_b^h}\big) \\
&\quad \le C\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\}.
\end{aligned} \tag{3.35}$$

Next, let $\boldsymbol{T}_h : [L^2(\Omega)]^2 \to \{\boldsymbol{r} \in [L^2(\Omega)]^2 \mid \boldsymbol{r}_{|K} \in [P_{k-2}(K)]^2 \ \ \forall K \in \mathcal{C}_h\}$ be the L^2-projection. Then, by (3.6) it holds

$$(\boldsymbol{T}_h \nabla\psi, \boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}) = 0. \tag{3.36}$$

Therefore, we obtain

$$\begin{aligned}
(\nabla\psi_h, &\boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}) = (\nabla(\psi_h - \psi), \boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}) + (\nabla\psi - \boldsymbol{T}_h\nabla\psi, \boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}) \\
&\le \|\nabla(\psi - \psi_h)\|_0 \|\boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}\|_0 + (\nabla\psi - \boldsymbol{T}_h\nabla\psi, \boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}) \\
&\le C\|\nabla(\psi - \psi_h)\|_0 \|\boldsymbol{\eta}\|_1 + (\nabla\psi - \boldsymbol{T}_h\nabla\psi, \boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}) \\
&= C\|\nabla(\psi - \psi_h)\|_0 \|\boldsymbol{\eta}\|_1 + (\nabla\psi - \boldsymbol{T}_h\nabla\psi, \boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta})_{\Omega_i^h} \\
&\qquad + (\nabla\psi - \boldsymbol{T}_h\nabla\psi, \boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta})_{\Omega_b^h} \\
&\le C\|\nabla(\psi - \psi_h)\|_0 \|\boldsymbol{\eta}\|_1 + \|\nabla\psi - \boldsymbol{T}_h\nabla\psi\|_{0,\Omega_i^h} \|\boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}\|_{0,\Omega_i^h} \\
&\qquad + \|\nabla\psi - \boldsymbol{T}_h\nabla\psi\|_{0,\Omega_b^h} \|\boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}\|_{0,\Omega_b^h} \\
&\le C\|\nabla(\psi - \psi_h)\|_0 \|\boldsymbol{\eta}\|_1 + C h_i^k \|\nabla\psi\|_{k-1,\Omega_i^h} \|\boldsymbol{\eta}\|_{1,\Omega_i^h} \\
&\qquad + C h_b \|\nabla\psi\|_{0,\Omega_b^h} \|\boldsymbol{\eta}\|_{1,\Omega_b^h}.
\end{aligned} \tag{3.37}$$

From Theorem 2.1 and the estimates already proved we thus have

$$\sup_{\boldsymbol{\eta} \in \boldsymbol{V}_h} \frac{|(\nabla\psi_h, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_1} \le C\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\}. \tag{3.38}$$

The right hand side of (3.33) is then bounded by the right hand side in the asserted estimate (3.29).

Step 3. From (2.16) and (3.24), by using (2.13) and (3.21), we get

$$\|w - w_h\|_1 \le \|\boldsymbol{\beta} - \boldsymbol{R}_h\boldsymbol{\beta}_h\|_0. \tag{3.39}$$

Now, it holds

$$\begin{aligned}\|\boldsymbol{\beta} - \boldsymbol{R}_h\boldsymbol{\beta}_h\|_0 &= \|(\boldsymbol{\beta} - \boldsymbol{R}_h\boldsymbol{\beta}) + (\boldsymbol{\beta} - \boldsymbol{\beta}_h) + (\boldsymbol{I} - \boldsymbol{R}_h)(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_0 \\ &\le \|\boldsymbol{\beta} - \boldsymbol{R}_h\boldsymbol{\beta}\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_0 + \|(\boldsymbol{I} - \boldsymbol{R}_h)(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_0.\end{aligned} \tag{3.40}$$

By using Lemma 3.3 and the previous estimates we then get

$$\|w - w_h\|_1 \le C\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\}. \tag{3.41}$$

We have now proved the asserted estimate (3.29).

Step 4. The L^2-estimates for the deflection and the rotation are proven by adapting the usual duality technique (cf. [10], [12]) and using the regularity estimate (2.19). □

Remark 3.1. For the shear force the previous theorem gives the estimate

$$t\|\boldsymbol{q} - \boldsymbol{q}_h\|_0 \le C\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\}, \tag{3.42}$$

which is utilized in the analysis of the postprocessing method in [11]. Also the splitting $w = w^r + w^0$ for the deflection and the corresponding regularity result of Theorem 2.1, which we have not used here, are utilized in [11].

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