

A REMARK ON THE HILLE–YOSHIDA GENERATOR THEOREM

Jarmo Malinen



A REMARK ON THE HILLE–YOSHIDA GENERATOR THEOREM

Jarmo Malinen

Jarmo Malinen: *A remark on the Hille–Yoshida generator theorem*; Helsinki University of Technology Institute of Mathematics Research Reports A469 (2004).

Abstract: *It is well-known (and can be proved in a number of ways) that a densely defined, closed operator A generates a bounded C_0 -semigroup if (and only if) the Hille–Yoshida resolvent condition*

$$\|(s_j - A)^{-k}\| \leq \frac{M}{s_j^k} \quad (1)$$

holds for some positive and unbounded sequence $\{s_j\}_{j \geq 1}$. We give a novel and short “frequency domain” proof for the observation that the resolvent condition (1), indeed, is only required for such sequences $\{s_j\}_{j \geq 1}$. The proof is based on studying the analytic function $s \mapsto (I - A/s)^{-1}$ whose values are power bounded operators.

AMS subject classifications: 47D03, 47A10, 47A30

Keywords: Semigroup generator, Hille–Yoshida theorem, Gibsons theorem

Jarmo.Malinen@hut.fi

ISBN 951-22-7017-X

ISSN 0748-3143

Espoo, 2004

Helsinki University of Technology

Department of Engineering Physics and Mathematics

Institute of Mathematics

P.O. Box 1100, 02015 HUT, Finland

email:math@hut.fi <http://www.math.hut.fi/>

1 Introduction

Let X be a Banach space. Let $A : \text{dom}(A) \rightarrow X$ be a generator of a bounded C_0 -semigroup $\{T(t)\}_{t \geq 0}$, satisfying $\sup_{t > 0} \|T(t)\| \leq M < \infty$. Such operators are precisely the closed, densely defined operators that satisfy the Hille – Yoshida resolvent condition

$$\|(s - A)^{-k}\| \leq \frac{M}{s^k} \quad \text{for all } s > 0 \text{ and } k \geq 1. \quad (2)$$

A classical references to this result are, of course, [4] (K. Yoshida) and [2] (E. Hille and R. S. Phillips). Both of these references give the stronger version of this result, as quoted in the abstract of this paper. The purpose of this paper is to give a short “frequency domain”, “complex analysis” proof for the following theorem:

Theorem 1. *Let A be a densely defined (closed) operator with $s_0 \in \mathbb{R}_+ \cap \rho(A)$, and let $M < \infty$. If for $s = s_0$ we have*

$$\|(I - A/s)^{-k}\| \leq M \quad \text{for all } k \geq 1, \quad (3)$$

then $(0, s_0] \subset \rho(A)$ and (3) holds for all $s \in (0, s_0]$.

Indeed, suppose that the resolvent condition (2) is known *only* for all $s \in \{s_j\}_{j \geq 1}$, where $\lim_{j \rightarrow \infty} s_j = +\infty$. Then (2) holds for *all* $s > 0$ as a direct consequence of Theorem 1.

Note that Theorem 1 has a flavor of the Maximum Modulus Theorem. All other proofs of Theorem 1 (that we know of) are carried out by using “time domain” techniques. It is rather unusual in harmonic analysis to have two precise characterizations of a same phenomenon, one on “each side” of the Fourier transform¹. This is the main motivation for writing this paper.

2 Resolvent condition for power-bounded operators

The discrete semigroups are generated by power bounded operators T . For such operators, a resolvent characterization has been published in [1] (A. Gibson), and it was independently rediscovered in [3, Theorem 2.7.1] (O. Nevanlinna).

Proposition 1. *Let $T \in \mathcal{L}(X)$ and $C < \infty$. Then the following are equivalent:*

(i) $\sup_{j \geq 0} \|T^j\| \leq C,$

(ii) *for all $x > 1$ and $k \geq 1$*

$$\|T^k(x - 1)^k(x - T)^{-k}\| \leq C, \quad (4)$$

and

¹Note that the Parseval’s identity is a positive example of this.

(iii) there exists a (monotone increasing) sequence $\{x_j\}_{j \geq 1} \subset (1, \infty) \cap \rho(T)$, such that $x_j \rightarrow \infty$ and the estimates (4) hold for $x = x_j$ for all $j \geq 1$ and $k \geq 1$.

Proof. Assume (i). Then for all $x > 1$, we have, by the nonnegativity of all scalar terms in sums

$$\begin{aligned} \|T^k(x-1)^k(x-T)^{-k}\| &= \left(1 - \frac{1}{x}\right)^k \|T^k \left(I - \frac{T}{x}\right)^{-k}\| \\ &= \left(1 - \frac{1}{x}\right)^k \|T^k \sum_{j \geq 0} \binom{k+j-1}{j} \left(\frac{T}{x}\right)^j\| \\ &\leq \left(1 - \frac{1}{x}\right)^k \sup_{j \geq 0} \|T^j\| \sum_{j \geq 0} \binom{k+j-1}{j} \left(\frac{1}{x}\right)^j \\ &= C \left(1 - \frac{1}{x}\right)^k \left(1 - \frac{1}{x}\right)^{-k} = C. \end{aligned}$$

So the resolvent condition in claim (ii) follows. The implication (ii) \Rightarrow (iii) is trivial. The final implication (iii) \Rightarrow (i) just by taking the limit as $x_j \rightarrow \infty$ in the resolvent condition. \square

There is a slight generalization of this results, and we give it here even though it will not be needed in the proof of Theorem 1.

Corollary 1. *Let $\alpha \in [0, 1)$ and $T \in \mathcal{L}(X)$. Then the powers of $T_\alpha := \alpha + (1-\alpha)T$ are bounded by constant C if and only if there exists a (monotone increasing) sequence $\{y_j\}_{j \geq 1} \subset (1, \infty) \cap \rho(T)$, such that $y_j \rightarrow \infty$ and the estimates*

$$\|T_\alpha^k(y_j-1)^k(y_j-T)^{-k}\| \leq C \quad (5)$$

hold for all $k \geq 1$.

Moreover, an operator $V \in \mathcal{L}(X)$ is power bounded by constant C if and only if there exists $\alpha \in [0, 1)$ and a (monotone increasing) sequence $\{y_j\}_{j \geq 1} \subset (1, \infty) \cap \rho(V_\alpha)$, such that $y_j \rightarrow \infty$ and the estimates

$$\|V^k(y_j-1)^k(y_j-V_\alpha)^{-k}\| \leq C \quad (6)$$

hold for all $k \geq 1$, where $V_\alpha := (V - \alpha) / (1 - \alpha)$.

Proof. For all $\alpha \neq 1$ and $x \in (1, \infty) \cap \rho(T_\alpha)$ we have

$$\begin{aligned} (x-1)(x-T_\alpha)^{-1} &= (x-1)(x-\alpha-(1-\alpha)T)^{-1} \\ &= \frac{x-1}{1-\alpha} \left(\frac{x-\alpha}{1-\alpha} - T\right)^{-1} = (y-1)(y-T)^{-1}, \end{aligned}$$

where $y = y(x) := (x-\alpha)(1-\alpha)^{-1}$ or, equivalently, $x = x(y) = \alpha + (1-\alpha)y$.

Assume that T_α is power-bounded by C . Then by implication (i) \Rightarrow (ii) of Proposition 1, we have for all $x > 1$ (and hence, because $\alpha \in [0, 1)$, for all $y > 1$)

$$\|T_\alpha^k(y-1)^k(y-T)^{-k}\| = \|T_\alpha^k(x-1)^k(x-T_\alpha)^{-k}\| \leq C$$

where $k \geq 1$ is arbitrary. This estimate holds in particular for any sequence $\{y_j\}_{j \geq 1}$ converging to ∞ , and the one direction of the first equivalence is now proved.

Conversely, assume that estimate (5) holds for all $k \geq 1$ and some sequence $\{y_j\}_{j \geq 1}$, having the stated properties. Define another sequence $\{x_j\}_{j \geq 1}$, by setting $x_j := \alpha + (1 - \alpha)y_j$. Because $\alpha < 1$, this new sequence satisfies the same conditions that have been imposed on $\{y_j\}_{j \geq 1}$. Now, for all $j \geq 1$, we have the estimates

$$\|T_\alpha^k(x_j-1)^k(x_j-T_\alpha)^{-k}\| = \|T_\alpha^k(y_j-1)^k(y_j-T)^{-k}\| \leq C$$

where $k \geq 1$ is arbitrary. Now implication (iii) \Rightarrow (i) of Proposition 1 gives the power-boundedness of T_α .

Let us proceed to prove the second equivalence. Fix $\alpha \in [0, 1)$ arbitrarily. Define $T := (V - \alpha)/(1 - \alpha)$. Then $T_\alpha = V$ and the power-boundedness of V is seen to be equivalent to the resolvent condition (6), by the first part of this corollary. \square

3 Proof of Theorem 1

Now begins the real fun, and we give the promised proof of Theorem 1.

Define for all $s \in \rho(A)$ the operator-valued function $T(s) := (I - A/s)^{-1}$. By the assumption of Theorem 1, $\sup_{k > 1} \|T(s_0)^k\| =: M < \infty$. Applying Proposition 1 shows that for all $x > 1$ and integers $k > 1$

$$\|T(s_0)^k(x-1)^k(x-T(s_0))^{-k}\| \leq M; \tag{7}$$

in particular such $x \in \rho(T(s_0))$. But now for all $x > 1$

$$\begin{aligned} T(s_0)(x-1)(x-T(s_0))^{-1} &= (x-1) \left(I - \frac{A}{s_0} \right)^{-1} \left(x - \left(I - \frac{A}{s_0} \right)^{-1} \right)^{-1} \\ &= (x-1) \left(x \left(I - \frac{A}{s_0} \right) - I \right)^{-1} = \left(I - \frac{A}{(1-1/x)s_0} \right)^{-1}. \end{aligned}$$

Denoting $s = (1 - 1/x)s_0$ we see from (7) that $\|(I - A/s)^{-k}\| \leq M$ for all such s . Because $x > 1$ was arbitrary, this estimate holds for all $s \in (0, s_0)$, thus proving Theorem 1.

References

- [1] A. G. Gibson. A discrete Hille – Yoshida theorem. *Journal of Mathematical Analysis and Applications*, 39:761–770, 1972.
- [2] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society, 1957.
- [3] O. Nevanlinna. *Convergence of Iterations for Linear Equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, Boston, Berlin, 1993.
- [4] Kōsaku Yosida. *Functional analysis*. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

(continued from the back cover)

- A460 Timo Eirola , Jan von Pfaler
Numerical Taylor expansions for invariant manifolds
April 2003
- A459 Timo Salin
The quenching problem for the N-dimensional ball
April 2003
- A458 Tuomas Hytönen
Translation-invariant Operators on Spaces of Vector-valued Functions
April 2003
- A457 Timo Salin
On a Refined Asymptotic Analysis for the Quenching Problem
March 2003
- A456 Ville Havu , Harri Hakula , Tomi Tuominen
A benchmark study of elliptic and hyperbolic shells of revolution
January 2003
- A455 Yaroslav V. Kurylev , Matti Lassas , Erkki Somersalo
Maxwell's Equations with Scalar Impedance: Direct and Inverse Problems
January 2003
- A454 Timo Eirola , Marko Huhtanen , Jan von Pfaler
Solution methods for R-linear problems in C^n
October 2002
- A453 Marko Huhtanen
Aspects of nonnormality for iterative methods
September 2002
- A452 Kalle Mikkola
Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati
Equations
October 2002

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at <http://www.math.hut.fi/reports/> .

- A470 Lasse Leskelä
Stabilization of an overloaded queueing network using measurement-based admission control
March 2004
- A464 Ville Turunen
Sampling at equiangular grids on the 2-sphere and estimates for Sobolev space interpolation
November 2003
- A463 Marko Huhtanen , Jan von Pfaler
The real linear eigenvalue problem in C^n
November 2003
- A462 Ville Turunen
Pseudodifferential calculus on the 2-sphere
October 2003
- A461 Tuomas Hytönen
Vector-valued wavelets and the Hardy space $H^1(\mathbb{R}^n; X)$
April 2003

ISBN 951-22-7017-X
ISSN 0748-3143
Espoo, 2004