Helsinki University of Technology Institute of Mathematics Research Reports
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Espoo 2002 A447

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Timo Salin: *Quenching-rate estimate for a reaction diffusion equation with weakly singular reaction term*; Helsinki University of Technology Institute of Mathematics Research Reports A447 (2002).

Abstract: *We study the quenching problem for the reaction diffusion equation $u_t - u_{xx} = f(u)$ with Cauchy-Dirichlet data, in the case where the reaction term is singular at $u = 0$ in the sense that $\lim_{u\downarrow 0} f(u) = -\infty$. For $u > 0$ we take $f(u)$ to be smooth and to satisfy $(-1)^k f^{(k)}(u) < 0$; $k = 0, 1, 2$. Furthermore, we assume that $f(u)$ is weakly singular (in a neighborhood of the origin) in the sense that: $|u^n f^{(n)}(u)| = o(|f(u)|)$, as $u \downarrow 0$, $n = 1$ and $n = 2$.*
We first show that for sufficiently large domains of x quenching occurs in finite time for these equations. The main result of this paper concerns the asymptotic behavior of the solution in a neighborhood of a quenching point. This result gives a uniform quenching rate-estimate in a region $|x| < C\sqrt{T-t}$ for the problem, when $(0, T)$ is a quenching point.

AMS subject classifications: 35K55, 35K60, 35B40

Keywords: Reaction-diffusion equation, quenching, quenching set, quenching rate, blow-up

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ISBN 951-22-5945-1
ISSN 0784-3143
Institute of Mathematics, Helsinki Univ. of Tech., Espoo, 2002

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1 Introduction

Consider the nonlinear diffusion problem

$$\begin{aligned} u_t - u_{xx} = f(u), \qquad x \in (-l, l), \quad t \in (0, T), \\ u(x, 0) = u_0(x), \quad x \in [-l, l], \\ u(\pm l, t) = 1, \quad t \in [0, T), \end{aligned} \tag{1.1}$$

where the initial function satisfies $0 < u_0(x) \leq 1$ and $u_0(\pm l) = 1$. Here T and l are positive constants. We assume that the reaction term $f(u)$ is singular at $u = 0$ in the sense that $\lim_{u \downarrow 0} f(u) = -\infty$. For $u > 0$ we take $f(u)$ to be smooth and to satisfy $(-1)^k f^{(k)}(u) < 0$; $k = 0, 1, 2$.

This type of reaction diffusion equation with singular reaction term arises in the study of electric current transients in polarized ionic conductors [17]. The problem can also be considered as a limiting case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics [7, 21].

The equation (1.1) has been extensively studied under assumptions implying that the solution $u(x, t)$ approaches zero in finite time. The reaction term then tends to infinity and the smooth solution ceases to exist. This phenomenon is called quenching. We say that a is a quenching point and T is a quenching time for $u(x, t)$, if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to a$ and $t_n \uparrow T$, such that $u(x_n, t_n) \to 0$ as $n \to \infty$.

In most of the papers that deal with the quenching problem for the equation (1.1), the reaction term is a power singularity i.e. $f(u) = -u^{-p}, \quad p > 0$. In this case it is well-known that for sufficiently large l quenching occurs in finite time [1, 2, 19]. It is also known that the set of quenching points is finite [13]. See also detailed review articles [18, 20].

An interesting field of inquiry concerning (1.1) with a power singularity has been the analysis of the local asymptotics of the solution as $t \uparrow T$ in the neighborhood of the quenching point. In particular, it has been shown that the quenching-rate satisfies

$$\lim_{t \uparrow T} u(x, t)(T - t)^{-1/(1+p)} = (1 + p)^{1/(1+p)}, \tag{1.2}$$

uniformly, when $|x - a| < C\sqrt{T - t}$ for every $C \in (0, \infty)$. This result was first established by Guo [13] for $p \geq 3$, and subsequently generalized to $p \geq 1$ by Fila and Hulshof [5]. For the weaker singularity $0 < p < 1$, (1.2) has been shown in [15]. The result (1.2) for higher dimensions has been obtained in [14] (when $p \geq 1$) and in [6] for the cases $p > 0$.

In [22] the equation (1.1) was studied in the case where we have only a logarithmic singularity, i.e., $f(u) = \ln(\alpha u), \quad \alpha \in (0, 1)$. It was shown there that despite of this weakening of the singularity, quenching still occurs for sufficiently large l and that the set of quenching points is finite. The main result in [22] concerns the asymptotic behavior of the solution in a neighborhood of a quenching point. In particular, it was shown that the quenching-rate satisfies

$$\lim_{t\uparrow T}\left(1+\frac{1}{T-t}\int_0^{u(x,t)}\frac{d\tau}{f(\tau)}\right)=0 \tag{1.3}$$

uniformly, when $|x-a|<C\sqrt{T-t}$ for every $C\in(0,\infty)$.

Note that (1.2) and (1.3) are equivalent if we substitute $f(u)=-u^{-p}$ in (1.3).

The proof of (1.3) is not based on earlier results on quenching. It uses similarity variables and energy estimates. This method of proof is different from that of earlier approaches where the corresponding quenching-rate estimate (1.2) has been obtained. (see Giga-Kohn [11, 12], Bebernes-Eberly [3], Guo [13, 14, 15])

In this paper the purpose is to extend the result (1.3) to a wider class of weak singularities. More precisely, we assume that $f(u)$ satisfies

$$|u^n f^{(n)}(u)|=o(|f(u)|),\quad n=1,2, \tag{1.4}$$

as $u\downarrow 0$. Furthermore we define $\tilde{f}(s)=-e^s\cdot\frac{f(e^{-s})}{f'(e^{-s})}$, and assume that

$$\tilde{f}(s(1+o(1)))=(1+o(1))\tilde{f}(s), \tag{1.5}$$

as $s\to\infty$. This definition means that for $a(s)\to 0$, as $s\to\infty$ there is $b(s)\to 0$, as $s\to\infty$ such that $\tilde{f}(s(1+a(s))=(1+b(s))\tilde{f}(s)$, as $s\to\infty$. Note that (1.4) implies $\tilde{f}(s)\to\infty$, as $s\to\infty$.

For example the reaction terms $f(u)=-|\ln(u)|^p,\quad p>0$ or $f(u)=-\ln(|\ln|\ln|\cdots(|\ln(u)|)\cdots|||)$ satisfy the conditions (1.4) and (1.5) (in a neighborhood of the origin). Note also that the conditions (1.4) and (1.5) hold for the reaction terms: $f(u)=-|\ln(u)|^p-|\ln(u)|^q$, $(p,q>0)$, $f(u)=-|\ln(u)|^p-\ln(|\ln(u)|)$, $(p>0)$, and $f(u)=-\ln(|\ln(u)|)-\ln(\ln(|\ln(u)|))$. Stronger singularities, like $f(u)=-u^{-p},\quad p>0$ or $f(u)=-\frac{u^{-p}}{|\ln(u)|},\quad p>0$, do not satisfy (1.4).

The main result of this paper is (1.3) for the equations (1.1) with the conditions (1.4) and (1.5). We formulate this in Theorem 3.1. The proof is further commented and given in Section 3. Before that we introduce some preliminary material in Section 2.

2 Preliminary results

We first study the possibility of quenching in finite time. There are two reasons why this may not occur. In the first case, we may have $u(x,t)\geq c>0$, for all $t>0$. This means that there is a solution to the corresponding stationary equation, which is a subsolution of the equation (1.1). On the other hand, it might happen that $u(x,t)>0$ for all t and x, but that $\min u(x,t)\to 0$, as $t\to\infty$. This second case is called quenching in infinite time.

Because $f(u)$ is a locally Lipschitz continuous function, we apply Theorem 1 (c) and Theorem 2 (c) in [2] (see the conclusion on p. 7 there), to conclude that the solution $u(x,t)$ of (1.1) quenches in finite time for l sufficiently large, provided that the similar stationary equation does not have a solution. We show below, that this last fact holds for the equations that have singularities satisfying (1.4) and (1.5).

Theorem 2.1. *Suppose that $u(x,t)$ is the solution of the equation* (1.1) *and that the reaction term satisfies* (1.4) *and* (1.5). *Then for l sufficiently large, $u(x,t)$ quenches in finite time.*

Proof. The solution $u(x,t)$ with $u_0 \in (0,1]$ is a subsolution of (1.1) with $u_0 \equiv 1$. Therefore by [2] it is sufficient to show that the corresponding stationary equation to the equation (1.1) does not have a solution that $u \in (0,1]$. Suppose that $u(x)$ is this solution. We follow the idea of [16].

By [10] we know that this solution is symmetric with respect to the origin and that $u'(0) = 0$. Therefore

$$\begin{gathered} u''(x) + f(u(x)) = 0, \qquad x \in (0,l), \\ u(l) = 1, \qquad u'(0) = 0. \end{gathered}$$

Substituting $v(z) = u(x) - 1$, where $z = x/l$, we get

$$\begin{gathered} v''(z) + l^2 f(v(z)+1) = 0, \qquad z \in (0,1), \\ v(1) = 0, \qquad v'(0) = 0. \end{gathered} \tag{2.1}$$

The corresponding (linear) eigenvalue problem is

$$\begin{gathered} u_n''(z) = -\lambda_n u_n(z), \qquad z \in (0,1), \\ u_n(1) = 0, \qquad u_n'(0) = 0. \end{gathered} \tag{2.2}$$

The eigenvalues and eigenfunctions of (2.2) are $\lambda_n = \pi^2(\frac{1}{2}+n)^2$ and $u_n(z) = \cos(\sqrt{\lambda_n} z), \quad n = 0,1,2,...$

We define the inner product by $\langle f,g\rangle = \int_0^1 f(z)g(z)dz$. From (2.1) and (2.2) it follows that

$$\langle v'', u_n\rangle + l^2\langle f(v(z)+1), u_n\rangle = \langle v, u_n''\rangle + l^2\langle f(v(z)+1), u_n\rangle = 0.$$

Therefore

$$-\lambda_n\langle v, u_n\rangle + l^2\langle f(v+1), u_n\rangle = 0.$$

Take $n = 0$. Then $u_0 > 0$, when $z \in (0,1)$, and $v \in (-1,0]$. Hence

$$l^2 = \frac{\pi^2}{4}\frac{\langle v, u_0\rangle}{\langle f(v+1), u_0\rangle} \le M < \infty.$$

The claim follows from this by choosing l large enough. □

We also know that quenching cannot occur on the boundary:

Theorem 2.2. [4] *The set of quenching points is a compact subset of $(-l, l)$.*

We can further apply [4] to conclude Theorem 2.3 concerning the asymptotics of the solution $u(x,t)$. This Theorem gives a lower bound as a function of t for $u(x,t)$ ($x \in (-\varepsilon + a, \varepsilon + a)$), when the quenching point is approached. It also gives an upper bound at a minimum point with respect to the x-variable.

The main Theorem improves this Theorem by giving a pointwise asymptotic behavior of $u(x,t)$ in the region $|x| < C\sqrt{T-t}$, when the quenching point is approached.

From now on we assume that the initial function satisfies

$$u_0''(x) + f(u_0(x)) \leq 0. \tag{2.3}$$

We can see by the maximum principle that this technical assumption guarantees that $u(x,t)$ is decreasing in time.

In Theorem 2.3 below we need the following definition. Blow-up means that a solution approaches infinity in finite time. We say that b is a blow-up point and T is a blow-up time for $v(x,t)$, if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to b$ and $t_n \uparrow T$, such that $v(x_n, t_n) \to \infty$ as $n \to \infty$.

Theorem 2.3. *Suppose that* (2.3) *holds and that quenching occurs at* $t = T$. *Then there exist positive constants* β,l_1 *and* t_1 *such that*

(a) $u_t - \beta f(u) \leq 0$, *when* $x \in (-l_1, l_1)$ *(the quenching points belong to this interval) and* $t \in [t_1, T)$.

(b) u_t *blows up, when* u *quenches.*

(c) $u_t(\underline{x}, t) - f(u(\underline{x}, t)) \geq 0$, *when* $t \in (0, T)$, *and* $\underline{x}$ *is a local minimum point of* $u(x,t)$ *with respect to* x.

Proof. Because we know that quenching happens (Theorem 2.1) and that the set of quenching points is bounded away from the boundary (Theorem 2.2), we can apply ([4], p. 1053-1054) to conclude the claim (a).

The item (b) follows directly from (a).

By the local existence theorem ([23] p.34) $u_{xx}(\underline{x}, t) \geq 0$, where $\underline{x}$ is a local minimum point of $u(x,t)$ with respect to x, and the claim (c) follows. □

3 Quenching-rate estimate

The main Theorem of this paper is Theorem 3.1. We first formulate the equation (1.1) with respect to new variables s, y and $w(y,s)$ in the equation (3.2), and then study the asymptotics of a quenching point by this equation. We assume that the initial function is symmetric in the sense that u_0 is even and $u_0'(r) \geq 0$. Then we know by the uniqueness of the solution and by the maximum principle that the solution is also symmetric. By choosing l large enough in (1.1), we can see by Theorem 2.1 and the symmetry of $u(x,t)$ that $(0,T)$ is (at least) a quenching point for some $T < \infty$.

We now begin to study the local asymptotics of the solution as a quenching point is approached. Define new variables:

$$y = \frac{x}{\sqrt{T-t}}, \qquad s = -\ln(T-t),$$

where $x \in [-l, l]$, $t \in [0, T)$, $y \in [-le^{\frac{1}{2}s}, le^{-\frac{1}{2}s}]$ and $s \in [-\ln T, \infty)$. Note that to analyse the asymptotic behavior as $t \uparrow T$ and $|x| \leq C\sqrt{T-t}$ corresponds the situation where $s \to \infty$ and $|y| \leq C$.

The function w is defined by

$$w(y, s) = 1 + \frac{1}{T-t}\int_0^{u(x,t)} \frac{d\tau}{f(\tau)}. \tag{3.1}$$

The equation (1.1) can now be written in the form

$$w_s = w_{yy} - \frac{1}{2}yw_y + w + F, \tag{3.2}$$

where $F = \frac{u_x^2}{(f(u))^2}f'(u)$, and $(y, s) \in (-le^{\frac{1}{2}s}, le^{\frac{1}{2}s}) \times (-\ln T, \infty)$.

The initial and boundary conditions are now

$$w(y, -\ln T) = 1 + \frac{1}{T}\int_0^{u_0(y\sqrt{T})} \frac{d\tau}{f(\tau)}, \quad w(\pm le^{\frac{1}{2}s}, s) = 1 + e^s\int_0^1 \frac{d\tau}{f(\tau)},$$

where $y \in [-le^{-\frac{1}{2}s}, le^{-\frac{1}{2}s}]$ and $s \geq -\ln T$.

Remark

Note that in the transformed equation (3.2) the nonlinear term F cannot be expressed explicitly as a function of y, s and w. For this reason, in the following, both the variables x and s, or y and t, may sometimes appear in the same equation. Another reason for this procedure is that in some cases it simplifies notations.

The goal is to prove:

Theorem 3.1. *Assume that u_0 is even, $u_0'(r) \geq 0$, $u_0(x) \in (0, 1]$ and (2.3) holds. We assume that the reaction term $f(u)$ is singular at $u = 0$ in the sense that $\lim_{u\downarrow 0} f(u) = -\infty$, and that the conditions (1.4) and (1.5) hold. For $u > 0$ we take $f(u)$ to be smooth and to satisfy $(-1)^k f^{(k)}(u) < 0$; $k = 0, 1, 2$. Let $u(x, t)$ be the corresponding solution of (1.1). Assume that $u(x, t)$ quenches at $(0, T)$ for some $T < \infty$. Then for any positive constant C,*

(a) $w(y, s) - w(0, s)(1 - \frac{1}{2}y^2) \to 0$, as $s \to \infty$ uniformly with respect to $|y| \leq C$, and

(b) $w(0, s) \to 0$, when $s \to \infty$.

Comment on the proof of the Theorem 3.1 The proof of (a) is built on Lemmas 3.2–3.8 and Corollary 3.9. The statement (b) follows from (a) and from Lemmas 3.10–3.14.

The first fundamental fact in the proof of (a) is to obtain the limit of Theorem 3.1 (a) in a weak sense (Lemma 3.4(b)). We observe that $F \to 0$ uniformly on compact y-intervals for the equation (3.2) (Lemma 3.2). Therefore, on compact y-intervals the equation $h'' - \frac{1}{2}yh' + h = 0$ can be considered as the stationary equation for (3.2), when s is large. A particular solution of this equation is $h_2(y) = (1 - \frac{1}{2}y^2)$. Both Lemmas 3.2 and 3.3 are essential in the proof of Lemma 3.4.

The second crucial ingredient in the proof of (a) is to realize that

$$\limsup_{s\to\infty}(w(y,s) - w(0,s)(1 - \frac{1}{2}y^2)) \leq 0,$$

uniformly for bounded y. This is a consequence of Lemmas 3.6 and 3.8. The argument of Theorem 3.1 (a) is based on this uniform upper bound, weak convergence (Lemma 3.4(b)) and the estimates of w (Lemma 3.3). Lemma 3.5 is needed in the proof of Lemma 3.6, and Lemma 3.7 is needed in the proof of Lemma 3.8.

The idea of the part (2) (the proof of Theorem 3.1 (b)) is to conclude the claim from the properties of the nonlinear term F of the equation (3.2), as $s \to \infty$. It is known by the part (1), that for large s the solution is formally $w(y,s) \approx w(0,s)h_2(y)$ (for bounded y), and then $\mathcal{L}w = w_{yy} - \frac{1}{2}w_y + w \approx 0$. We expect that the nonlinear part will eventually dominate the linear part in the equation (3.2). Concerning the reaction term F it is known , that it is zero only at the point $y = 0$ and otherwise positive. Thus for a large s^*, the reaction term F does not have any contribution on $w(0,s)$, but for a large y it has a small increasing contribution on $w(y,s)$. More precisely, for large s the reaction term behaves formally as $F \approx f_1(y,s)f_2(s)$, where $\frac{\partial}{\partial y}f_1(y,s) \approx w_y^2 \geq 0$ (by the part (a)) and $f_2(s) \approx f'(\underline{u})(T-t) \to 0$. Therefore, somewhat later ($s = s^* + \delta$), the profile of the solution $w(y,s)$ is formally $w(0,s)[h_2(y) + g(s)\epsilon(y)]$, where the function $\epsilon(y)$ is non-decreasing ($y \geq 0$) and $\epsilon(0) = 0$. Because the solution $w(y,s)$ has to preserve the asymptotical form obtained in the part (1), the only possibility is, that $w(0,s)$ has to be decreasing, and that the limit value has to be zero.

The equation (3.2) is studied as a dynamical system in the space $L^2_\rho(R)$, where $\rho(y) = \exp(-\frac{y^2}{4})$. Then the eigenvalues and eigenfunctions of the operator $\mathcal{L}$ are well-known. The scaled Hermite polynomials form an orthonormal base on that space, and the eigenvalue of the second order polynomial $h_2(y)$ is zero. By the part (1), it is known that this polynomial is dominant, as $s \to \infty$. So we obtain that the multiplier function $a_2(s)$ of h_2 in the Fourier expansion of the function w is asymptotically equal to $w(0,s)$ (Lemma 3.12), and that $a_2(s) \to 0$ (see (3.46)). To obtain (3.46), we first write the equation (3.2) in the form (3.40). Then we conclude by Lemmas 3.12, 3.13, 3.14, and part (a) that the first three terms on the right-hand side of this equation converge to zero, and that the last term leads to (3.46).

The domain of the solution $w(y,s)$ of the equation (3.2) is not (with respect to y) the whole R. Therefore, the above properties of L^2_ρ and $\mathcal{L}$ cannot be applied directly to the equation (3.2). This difficulty can be avoided by first extending the equation (1.1) to all $x \in R$, and observing that the solution of this equation in the region $\{(x,t) \in R^2 | x \in (-l,l), t \in (0,T)\}$ is the same as the solution of the original equation (1.1). Then the transformed solution $\tilde{w}(y,s)$ corresponding to the extended solution $\tilde{u}(x,t)$ is also defined for all $y \in R$.

The assumption (1.4) is needed in Proof of Theorem 3.1 (a), and thus also in Proof of (b). But the assumption (1.5) is used only in the part (b).

Before we start to prove Theorem 3.1, we emphasize that the following analysis does not exclude the possibility of quenching on an entire interval. In fact we can quarantee that the set of quenching points is finite for the equation (1.1) that satisfies conditions (1.4) and (1.5) only for $f(u) = \ln(\alpha u)$ by [22]. We don't know, can a further weakening of singularity in (1.1) lead to quenching on a whole interval. However, by Theorem 2.2, quenching points are bounded away from the boundary, and this observation is sufficient in what follows.

3.1 Proof of Theorem 3.1 (a)

We begin by introducing the inequalities (3.3), (3.4) and (3.5), which will be useful in the sequel.

From (1.4) with $n = 1$ we can first verify that $f(u) \in L^1(0,1)$, and then by this fact that $\lim_{u \to 0} uf(u) = 0$. Therefore we can obtain by (1.4) ($n = 1$ and by partial integration that

$$\int_0^u f(\tau) d\tau = uf(u) + o(|\int_0^u f(\tau) d\tau|) \tag{3.3}$$

and

$$\int_0^u \frac{d\tau}{f(\tau)} = \frac{u}{f(u)} + o(|\int_0^u \frac{d\tau}{f(\tau)}|), \tag{3.4}$$

as $u \to 0$.

By Theorem 2.3 (a) and (c) we have the inequalities $\underline{u}_t - f(\underline{u}) \geq 0$ and $u_t - \beta f(u) \leq 0$, in $(x,t) \in (-\delta, \delta) \times (T - \delta, T)$. Hence there is $C > 0$ such that

$$-\int_0^{\underline{u}} \frac{d\tau}{f(\tau)} \leq T - t \leq -C \int_0^{\underline{u}} \frac{d\tau}{f(\tau)} \leq -C \int_0^u \frac{d\tau}{f(\tau)}. \tag{3.5}$$

Lemma 3.2.

(a) $u(x,t) \to 0$ uniformly, when $t \uparrow T$ and $|x| \leq C\sqrt{T-t}$.

(b) F is uniformly bounded, when $(x,t) \in [-l,l] \times [0,T)$.

(c) $F \to 0$ uniformly, when $t \uparrow T$ and $|x| \leq C\sqrt{T-t}$.

Proof. By Theorem 2.1, the conditions (2.3), (1.4) (when $n = 1$) and (3.3) we can see that to conclude this Lemma it is sufficient to show the inequality:

$$P(x,t) \stackrel{def}{=} \frac{1}{2} u_x(x,t)^2 + \int_{u(0,t)}^{u(x,t)} f(\tau) d\tau \leq 0, \tag{3.6}$$

in the region $A = \{(x,t)|x \in (-l,l), t \in (0,T)\}$.

Because the initial function u_0 is symmetric, then $u_x(0,t) = 0$ for all $t \in [0,T)$, and thus $P(0,t) = 0$ for all $t \in [0,T)$. Differentiating the function P with respect to x, we get $P_x = u_x u_{xx} + u_x f(u) = u_x u_t$, by the equation (1.1). It follows from the symmetry that $P_r \leq 0$ ($r = |x|$). So we have obtained the inequality (3.6). □

Lemma 3.3. *There exist positive constants c_1,c_2 and δ, independent of y and s, such that for all $s \geq -\ln T$,*

(a) $-c_1 \leq w_{yy}(0,s) \leq 0$.

(b) $-c_2 \leq w_{yy}(y,s)$, *when* $-le^{\frac{1}{2}s} \leq y \leq le^{\frac{1}{2}s}$.

(c) $-c_2 y \leq w_y(y,s) \leq 0$, *when* $0 \leq y \leq le^{\frac{1}{2}s}$. *Furthermore,* $0 \leq w_y(y,s) \leq -c_2 y$, *when* $-le^{\frac{1}{2}s} \leq y \leq 0$.

(d) $0 \leq w(0,s) \leq 1-\delta$.

(e) $-\frac{1}{2}c_2 y^2 \leq w(y,s) \leq 1-\delta$, *when* $-le^{\frac{1}{2}s} \leq y \leq le^{\frac{1}{2}s}$.

Proof. The claims can be verified from the symmetry of the solution, Definition (3.1), Theorem 2.3(a),(c) and Lemma 3.2. See the detailed calculations in [22]. □

Lemma 3.4. *Let $\rho(y) = \exp(\frac{-y^2}{4})$, and let $a(s)$ be a bounded function for $s \geq -\ln T$. Let $h_2(y) = (1-\frac{1}{2}y^2)$ be the second order Hermite-polynomial. Then*

(a) $\int_0^{le^{\frac{1}{2}s}} w(y,s)\rho(y)dy \to 0$, *when* $s \to \infty$.

(b) $\int_0^{le^{\frac{1}{2}s}} (w(y,s) - a(s)h_2(y))\rho(y)dy \to 0$, *when* $s \to \infty$.

Proof. Multiply the equation (3.2) by ρ to obtain

$$(w_s - w)\rho = (w_y\rho)_y + F\rho.$$

and define $I(s) = \int_0^{le^{\frac{1}{2}s}} w(y,s)\rho(y)dy$. Then we get

$$I'(s) - I(s) = (\frac{1}{2}le^{\frac{1}{2}s}w(le^{\frac{1}{2}s},s) + w_y(le^{\frac{1}{2}s},s))\rho(le^{\frac{1}{2}s}) + \int_0^{le^{\frac{1}{2}s}} F\rho dy. \quad (3.7)$$

Because, by Theorem 2.2, quenching on the boundary is impossible, it follows from the boundary conditions and Lemma 3.3 that the first two term on the right-hand side converge to zero in (3.7). Furthermore, $\int_0^{le^{\frac{1}{2}s}} F\rho dy \to 0$ by Lemma 3.2, and therefore $I'(s) - I(s) \to 0$, as $s \to \infty$. The statement (a) can be obtained from this. Suppose that for some $\varepsilon > 0$ there exists a sequence $\{s_i\}$ such that $|I(s_i)| \geq \varepsilon$. Then $|I(s_i)| \to \infty$, as $s_i \to \infty$. This contradicts Lemma 3.3.

The claim (b) follows from the item (a) and from partial integration:

$$\int_0^\infty h_2(y)\rho(y)dy = \int_0^\infty \rho(y)dy + (y(\rho(y)))(\infty) - \int_0^\infty \rho(y)dy = 0.$$

□

Lemma 3.5. *There exist constants $C \in (0, \infty)$, $l_1 \in (0, l)$ and $t_1 \in (0, T)$ such that $0 \leq -u_t(x,t) \leq -Cf(u(0,t))$, when $x \in [0, l_1]$ and $t \in [t_1, T)$.*

Proof. We first show that there exist constants $\gamma \in (0, \infty)$, $l_1 \in (0, l)$ and $t_1 \in (0, T)$ such that

$$u_{rt} - \gamma u_t \geq 0, \tag{3.8}$$

when $x \in [-l_1, l_1]$ and $t \in [t_1, T)$. Let $J(r,t) = u_{rt}(r,t) - \gamma u_t(r,t)$. Differentiating this, we get

$$J_t - J_{rr} - f'(u)J = u_r u_t f''(u).$$

The right-hand side of this equation is non-negative, by the facts that $u_r \geq 0$, $u_t < 0$ and $f''(u) < 0$. Because of Theorems 2.3 and 2.2, we can choose l_1 and t_1 such that $|u_{rt}| \leq M$ and $u_t \in [-c_1, -c_2]$ on the parabolic boundary of $[-l_1, l_1] \times [t_1, T)$. Then we can take γ sufficiently large such that $J \geq 0$ on that parabolic boundary. Now we obtain (3.8) from the maximum principle.

We can now conclude by (3.8) that

$$u_t(x,t) - u_t(0,t) = \int_0^x u_{tx}(\eta,t)d\eta \geq \gamma \int_0^x u_t(\eta,t)d\eta =$$
$$\gamma \int_0^x (u_{xx}(\eta,t) + f(u(\eta,t)))d\eta \geq \gamma(u_x(x,t) + xf(u(0,t))).$$

and because we know by Theorem 2.3 that $u_t(0,t) - f(u(0,t)) \geq 0$, then

$$-u_t(x,t) \leq -u_t(0,t) - \gamma x f(u(0,t)) \leq -f(u(0,t))(1+\gamma x),$$

and therefore $0 \leq -u_t(x,t) \leq -Cf(u(0,t))$.

□

Lemma 3.6. *For $0 \leq y \leq C$, we have $\limsup_{s\to\infty} w_{yyy}(y,s) \leq 0$ uniformly.*

Proof. Differentiating the definition (3.1), we get by (1.1) after some calculations that

$$w_{yyy} = \sum_{i=1}^{4} G_i,$$

where

$$G_1(x,t) = \sqrt{T-t}\frac{u_{xt}}{f(u)},$$
$$G_2(x,t) = -\sqrt{T-t}\frac{u_x u_t}{f(u)^2}f'(u),$$
$$G_3(x,t) = -2\sqrt{T-t}\frac{u_x}{f(u)}\Big(\frac{u_{xx}}{f(u)} - F\Big)f'(u),$$
$$G_4(x,t) = -\sqrt{T-t}\Big(\frac{u_x}{f(u)}\Big)^2 f''(u)u_x.$$

We shall prove that $\limsup_{t\uparrow T} G_1(x,t) \leq 0$, and that $G_i \to 0$ uniformly for bounded y, as $s \to \infty$ and $i = 2, 3, 4$.

1. i=1:

By (3.8), $\sqrt{T-t}\frac{u_{xt}}{f(u)} \le \gamma\sqrt{T-t}\frac{u_t}{f(u)}$, when $x>0$. So it suffices to show that $(T-t)\frac{u_t^2}{(f(u))^2} \to 0$, as $t \uparrow T$. From (1.4) (when $n=1$), (3.4), (3.5), Lemmas 3.2 and 3.5 it follows that

$$0 \le (T-t)\Big(\frac{u_t}{f(u)}\Big)^2 \le C\frac{u}{-f(\underline{u})}\frac{f(\underline{u})^2}{f(u)^2} \le C\frac{\underline{u}|f(\underline{u})|}{f(u)^2} \to 0.$$

2. i=2:

Using (3.1), we obtain $0 \le G_2 = -w_y(T-t)(-u_t)\frac{f'(u)}{-f(u)}$. Applying (3.4), (3.5), Lemmas 3.3 and 3.5, we get

$$0 \le G_2 \le C\frac{u}{-f(\underline{u})}(-f(\underline{u}))\frac{f'(u)}{-f(u)}.$$

The claim follows from (1.4) and Lemma 3.2(a).

3. i=3:

The function $G_3(x,t)$ can be written in the form:

$$G_3 = 2G_2 + 2w_y(1+F)(T-t)f'(u).$$

By the case i=2, Lemmas 3.2 and 3.3 it is sufficient show that $(T-t)f'(u) \to 0$. Using (3.4), (3.5) and (1.4), we deduce

$$0 \le (T-t)f'(u) \le C\frac{uf'(u)}{-f(u)} \to 0.$$

4. i=4:

By a simple modification we get $0 \le G_4 = -w_y\frac{u_x^2 f''(u)}{f(u)}(T-t)$. The inequality (3.6), the formulas (3.3) and (3.5) yield

$$0 \le G_4 \le -Cw_y\frac{u^2|f''(u)|}{|f(u)|}.$$

The claim follows from Lemmas 3.2 and 3.3, and (1.4) (when $n=2$). □

Lemma 3.7. *There exists a positive constant M such that $(T-t)u_{tt} \le M$ in some neighborhood $N=(-a,a)\times(T-\delta,T)$ of $(0,T)$.*

Proof. Let $H=(T-t)u_{tt}-M$, where the constant $M>0$ will be determined later. Then

$$H_t - H_{xx} - bH = (T-t)f''(u)u_t^2 + \frac{M}{T-t}\big(f'(u)(T-t)-1\big),$$

where $b = f'(u) - \frac{1}{T-t}$ is a locally bounded function. We can see as in the proof of Lemma 3.6 that $f'(u)(T-t) \to 0$, and therefore there exists a neighborhood $N=(-a,a)\times(T-\delta,T)$ of $(0,T)$, which is independent of M, such that

$$H_t - H_{xx} - bH \le 0,$$

on N. When M is chosen big enough, then, by Theorem 2.2, $H \le 0$ on the parabolic boundary of N. The claim now follows from the maximum principle. □

Lemma 3.8. *Let $w(y,s)$ be the solution of (3.2). Then* $\lim_{s\to\infty} w_s(0,s) = 0$.

Proof. Differentiating the equation (3.1), we get by (1.1) that

$$\begin{aligned} \frac{1}{T-t}\{w_{ss} + \frac{1}{2}yw_{ys} - w_s - \frac{1}{2}yw_y\} = -x\frac{\partial}{\partial t}\{\frac{u_x}{2(T-t)f(u)}\}+ \\ + \frac{u_{tt}}{f(u)} - \frac{u_t^2}{f(u)^2}f'(u). \end{aligned} \tag{3.9}$$

In (3.9), take $y = 0$ to obtain

$$w_{ss}(0,s) - w_s(0,s) = (T-t)\{\frac{u_{tt}(0,t)}{f(u(0,t))} - \frac{u_t(0,t)^2}{f(u(0,t))^2}f'(u(0,t))\}.$$

Here the last term of the right-hand side converges to zero by Theorem 2.3 (c) and by the fact that $(T-t)f'(u(0,t)) \to 0$. Thus, applying Lemma 3.7, we have

$$\liminf_{s\to\infty}\{w_{ss}(0,s) - w_s(0,s)\} \geq 0. \tag{3.10}$$

We prove next that $\liminf_{s\to\infty} w_s(0,s) \geq 0$.

By Lemma 3.6, for every $\varepsilon > 0$ and $C > 0$, there exists $s_1 > -\ln T$ such that $w_{yyy} < \varepsilon$, when $s \geq s_1$ and $0 \leq y \leq C < \infty$. Integrating this inequality three times with respect to y, we get

$$w_{yy}(y,s) - w_{yy}(0,s) < \varepsilon y, \tag{3.11}$$

$$w_y(y,s) - yw_{yy}(0,s) < \frac{1}{2}\varepsilon y^2, \tag{3.12}$$

$$w(y,s) - w(0,s) - \frac{1}{2}y^2 w_{yy}(0,s) < \frac{1}{6}\varepsilon y^3, \tag{3.13}$$

when $y \in [0,C]$. Because $w_s(0,s) = w_{yy}(0,s) + w(0,s)$, it follows from the inequality (3.13) that

$$\limsup_{s\to\infty}\{w(y,s) + w_{yy}(0,s)(1 - \frac{1}{2}y^2) - w_s(0,s)\} \leq 0, \tag{3.14}$$

uniformly, when $0 \leq y \leq C < \infty$. Let $g(y,s) = w(y,s) + w_{yy}(0,s)(1 - \frac{1}{2}y^2)$, and consider the function

$$G(s) = \int_0^{le^{\frac{1}{2}s}} (g(\eta,s) - w_s(0,s))\rho(\eta)d\eta.$$

An application of (3.14) to this definition gives that

$$\limsup_{s\to\infty} G(s) \leq 0. \tag{3.15}$$

We can see, by Lemmas 3.3 and 3.4 (b), that

$$G(s) + w_s(0,s)\int_0^{le^{\frac{1}{2}s}} \rho(\eta)d\eta \to 0, \tag{3.16}$$

as $s \to \infty$. Using the formulas (3.15) and (3.16), we get

$$\liminf_{s\to\infty} w_s(0,s) \geq 0. \tag{3.17}$$

Suppose now that for some $\varepsilon > 0$ there exists a sequence $s_i \to \infty$ such that $w_s(0,s_i) \geq \varepsilon$. From this it follows by (3.10) that $w_s(0,s_i) \to \infty$. This contradicts Lemma 3.3. Hence $\limsup_{s\to\infty} w_s(0,s) \leq 0$. The claim follows from this and from (3.17). □

After these preliminary Lemmas we turn to the proof of Theorem 3.1 (a). Let g be as in the proof of Lemma 3.8. By this Lemma, and by (3.14),

$$\limsup_{s\to\infty} g(y,s) \leq 0 \tag{3.18}$$

uniformly, when $0 \leq y \leq C < \infty$. Furthermore, by Lemma 3.4(b) (where $a(s) = -w_{yy}(0,s)$)

$$\lim_{s\to\infty} \int_0^{le^{\frac{1}{2}s}} g(\eta,s)\rho(\eta)d\eta = 0. \tag{3.19}$$

By Lemma 3.3 it holds that $|g_y| \leq C|y|$. Therefore it follows from (3.18), (3.19) and the symmetry of the solution that

$$\lim_{s\to\infty} g(y,s) = 0, \tag{3.20}$$

uniformly, when $|y| \leq C < \infty$. By Lemma 3.8, $w_{yy}(0,s) + w(0,s) \to 0$, as $s \to \infty$, and so Theorem 3.1 (a) follows from the equation (3.20).

Corollary 3.9. *For $|y| \leq C < \infty$, we have $\lim_{s\to\infty} w_s(y,s) = 0$ uniformly.*

Proof. By Lemma 3.8, $w_s(0,s) = w_{yy}(0,s) + w(0,s) \to 0$. Combining this with inequality (3.12), we have

$$\limsup_{s\to\infty} \{w_y(y,s) + yw(0,s)\} \leq 0, \tag{3.21}$$

uniformly, when $0 \leq y \leq C < \infty$. Writing

$$w(y,s) - w(0,s)(1 - \frac{1}{2}y^2) = \int_0^y (w_\eta(\eta,s) + \eta w(0,s))d\eta,$$

we obtain, by Theorem 3.1 (a), (3.21) and Lemma 3.3, that

$$\lim_{s\to\infty} (w_y(y,s) + yw(0,s)) = 0, \tag{3.22}$$

uniformly for bounded y. Correspondingly, we can conclude by (3.11),(3.22), Lemmas 3.3 and 3.8 that $\lim_{s\to\infty}(w_{yy}(y,s) + w(0,s)) = 0$. Finally we obtain the claim from this, (3.22), Theorem 3.1 (a) and Lemma 3.2(c). See the detailed calculations in [22]. □

3.2 Proof of Theorem 3.1(b)

We will now replace the equation (1.1) by an extended one, defined on the entire real line with respect to x. This equation of course admits the same solutions as (1.1) on the original interval $(-l, l)$. The technical construction is done similarly as in [24] or in [8]. Without loss of generality, we may assume that $l = 1$ in the equation (1.1). So let $x \geq 1$, and define the kernels:

$$V(x,t) = \frac{1}{\sqrt{\pi t}} \exp(-\frac{x^2}{4t}),$$

$$W(x,t) = \frac{x}{2\sqrt{\pi t^3}} \exp(-\frac{x^2}{4t}),$$

when $x \in R$ and $t > 0$.

Differentiating these, we can see that $V_x = -W$, $V_t = V_{xx}$ and $W_t = W_{xx}$.

Define the extension $\overline{u}(x,t)$ of $u(x,t)$, when $x \geq 1$ and $t > 0$ by

$$\overline{u}(x,t) = (x-1) \int_0^t W(x-1, t-\tau) u_x(1,\tau) d\tau + 1. \tag{3.23}$$

Here $u_x(1,t)$ is obtained from the equation (1.1) ($u_x(1,t) = \lim_{z\uparrow 1} u_x(z,t)$).

Lemma 3.10. *The function* $\overline{u}$ *satisfies:*

$$\overline{u}_t - \overline{u}_{xx} = 2u_x(1,0)V(x-1,t) + 2\int_0^t V(x-1,t-\tau)u_{x\tau}(1,\tau)d\tau,$$

when $x > 1$.

Proof. The claim follows directly by differentiating (3.23) □

Correspondingly in the extension of u to the left of $x = -1$ the term $u_x(1,t)$ in the equation (3.23) is replaced by the term $u_x(-1,t)$.

An extended equation is now defined

$$\tilde{u}_t - \tilde{u}_{xx} = f(\tilde{u}(x,t)); \qquad x \in R\backslash\{\pm 1\}, \quad 0 < t < T,$$

where

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & \text{when } |x| \leq 1, \\ \overline{u}(x,t), & \text{when } |x| > 1, \end{cases}$$

$$f(\tilde{u}) = \begin{cases} f(u), & \text{when } |x| \leq 1, \\ \overline{g}(x,t), & \text{when } |x| > 1, \end{cases}$$

and

$$\overline{g}(x,t) = 2u_x(1,0)V(x-1,t) + 2\int_0^t V(x-1,t-\tau)u_{x\tau}(1,\tau)d\tau. \tag{3.24}$$

We can see that $\tilde{u} \in C^1(R)$ (fixed t), but f is not continuous at $x = \pm 1$, and therefore $\tilde{u}$ is not twice continuously differentiable.

Because $u(x,t)$ cannot quench at $x = \pm 1$, then the functions $u_x(1,t)$ and $u_{xt}(1,t)$ are uniformly bounded.

Lemma 3.11. *The functions* $\overline{u}(x,t)$ *and* $\overline{g}(x,t)$ *satisfy*

$$1 \le \overline{u}(x,t) < c_1 < \infty,$$

and

$$0 \le \overline{g}(x,t) < c_2 < \infty,$$

when $|x| > 1$ *and* $0 < t < T$.

Proof. This Lemma can be obtained from (3.6), (3.23) and (3.24). □

From (3.23) one may also obtain

$$|\overline{u}_x| \le c_4 < \infty, \tag{3.25}$$

for $|x| > 1$ and $0 < t < T$.

Define, when $y \in R$ ($y = \frac{x}{\sqrt{T-t}}$) and $s > -\ln T$:

$$\tilde{w}(y,s) = 1 + \frac{1}{T-t}\int_0^{\tilde{u}(x,t)} \frac{d\tau}{f(\tau)}. \tag{3.26}$$

Differentiating the definition (3.26), we get

$$\tilde{w}_s - \tilde{w}_{yy} + \frac{1}{2}y\tilde{w}_y - \tilde{w} = \tilde{F}, \tag{3.27}$$

where $\tilde{F} = F$, when $y \in (-e^{\frac{1}{2}s}, e^{\frac{1}{2}s})$, and at intervals $(-\infty, -e^{\frac{1}{2}s})$ and $(e^{\frac{1}{2}s}, \infty)$:

$$\tilde{F} = \frac{\overline{g}(x,t)}{f(\overline{u})} - 1 + \frac{\overline{u}_x^2}{f(\overline{u})^2} f'(\overline{u}), \tag{3.28}$$

where $\overline{g}$ is defined by the equation (3.24) and f by the equation (1.1). Using (3.25), (3.26), (3.28) and Lemma 3.11, we obtain, when $|y| > le^{\frac{1}{2}s}$ and $s > -\ln T$,

$$|\tilde{w}(y,s)| \le C(y^2+1), \tag{3.29}$$

$$|\tilde{w}_y(y,s)| \le C|y|,$$

and

$$|\tilde{F}| \le M < \infty. \tag{3.30}$$

Consider now the extended equation (3.27) as a dynamical system in the space

$$L^2_\rho(R) = \{g \in L^2_{loc}(R) | \int_R g(y)^2 \rho(y) dy < \infty\}.$$

We use from now on the notation $w = \tilde{w}$. Then $w_s - \mathcal{L}w = F$, where $\mathcal{L}w = w_{yy} - \frac{1}{2}yw_y + w$, on the set $R \times [-\ln T, \infty)$.

The space L^2_ρ is a Hilbert space with an inner product

$$\langle f, g \rangle_{L^2_\rho} = \int_R f(y)g(y)\rho(y)dy.$$

Concerning the linear operator $\mathcal{L}$ it is known that (see [9]), it is self-adjoint, i.e., that

$$\langle \mathcal{L}f, g\rangle_{L^2_\rho} = \langle f, \mathcal{L}g\rangle_{L^2_\rho}, \tag{3.31}$$

with spectrum $\lambda_k = 1 - \frac{1}{2}k$; $\quad k = 0, 1, 2, \ldots$ The corresponding eigenfunctions are $\tilde{h}_k(y) = \alpha_k H_k(\frac{1}{2}y)$, where H_k are the (standard) Hermite polynomials and $\alpha_k = (\pi^{\frac{1}{2}} 2^{k+1} k!)^{-\frac{1}{2}}$. The first three eigenfunctions are

$$\tilde{h}_0 = \frac{1}{\sqrt{2}}\pi^{\frac{-1}{4}}, \quad \tilde{h}_1 = \frac{1}{2}\pi^{\frac{-1}{4}}y, \quad \tilde{h}_2 = \frac{1}{2}\pi^{\frac{-1}{4}}(\frac{1}{2}y^2 - 1).$$

The Fourier-expansion of w with respect to this base is:

$$w(y,s) = \sum_{k=0}^{\infty} \tilde{a}_k(s)\tilde{h}_k(y).$$

Then one has

Lemma 3.12. *Let* $a_2(s) = -\frac{1}{2}\pi^{\frac{-1}{4}}\tilde{a}_2(s)$. *Then* $a_2(s) - w(0,s) \to 0$, *as* $s \to \infty$.

Proof. Let $\phi(y,s) = w(y,s) - w(0,s)h_2(y)$, where h_2 is defined as in Lemma 3.4. Projecting the function ϕ to the subspace generated by the function $\tilde{h}_2$, we get

$$\langle \phi, \tilde{h}_2\rangle_{L^2_\rho} = \sum_k \tilde{a}_k(s)\langle \tilde{h}_k, \tilde{h}_2\rangle_{L^2_\rho} - w(0,s)\langle h_2, \tilde{h}_2\rangle_{L^2_\rho} = -2\pi^{\frac{1}{4}}(a_2(s) - w(0,s)).$$

Applying Hölder's inequality to this, it follows that

$$|a_2(s) - w(0,s)| \le C(\int_R (w(y,s) - w(0,s)h_2)^2\rho)^{\frac{1}{2}} \|h_2\|^{\frac{1}{2}}_{L^2_\rho} \to 0,$$

by Theorem 3.1(a) and the inequality (3.29). □

In the following we replace $-\ln(T-t)$ (used in the proof for $f(u) = \ln(\alpha u)$ in [22]) by $g(-\ln(T-t))$, where the function g is defined such that

$$\lim_{u\to 0} \frac{uf'(u)}{-f(u)} g(-\ln(u)) = 1. \tag{3.32}$$

This function satisfies

$$\lim_{s\to\infty} \int_K^s \frac{ds}{g(s)} = \infty. \tag{3.33}$$

The formula (3.33) can be deduced directly from (3.32) by writing first

$$g(s) = -(1 + o(1))e^s \cdot \frac{f(e^{-s})}{f'(e^{-s})},$$

as $s \to \infty$, and then integrating with respect to s. Note that (3.33) is a crucial property of g in the proof of Theorem 3.1 (b) (see the formulas (3.45) and (3.46)).

By the assumption (1.5) we can see that

$$g(s(1+o(1))) = g(s)(1+o(1)), \tag{3.34}$$

as $s \to \infty$.

Lemma 3.13. *The inequalities*

$$0 \le \; f'(u)(T-t)g(-\ln(T-t)) \le M < \infty$$

hold on the set $[-l, l] \times [0, T)$.

Proof. By (1.4) (when $n = 1$), (3.4) and (3.5) we conclude that

$$-\ln(T-t) = -\ln(\underline{u})(1+o(1)).$$

Recall that $f''(u) < 0$. Therefore we can estimate, by (3.4), (3.5) and (3.34), to obtain

$$0 \le f'(u)(T-t)g(-\ln(T-t)) \le Cf'(\underline{u})\frac{\underline{u}}{-f(\underline{u})}g(-\ln(\underline{u})).$$

The claim follows from (3.32). □

Lemma 3.14. *For the solution* $u(x,t)$ *one has*

$$f'(u(x,t))(T-t)g(-\ln(T-t)) - \frac{1}{1-w(0,s)h_2(y)} \to 0$$

uniformly for bounded y*, as* $t \uparrow T$ $(s \to \infty)$*.*

Proof. By Theorem 3.1 (a),

$$1 + \frac{1}{T-t}\int_0^u \frac{d\tau}{f(\tau)} - w(0,s)h_2(y) \to 0,$$

uniformly for bounded y, as $t \uparrow T$. Dividing this by the function $1 - w(0,s)h_2(y)$ ($\neq 0$ by Lemma (3.3)), we get

$$\frac{\int_0^u \frac{d\tau}{-f(\tau)}}{(T-t)(1-w(0,s)h_2(y))} \to 1, \tag{3.35}$$

uniformly for bounded y, as $t \uparrow T$. From the properties of the logarithmic function it follows that

$$\ln\Big(\frac{1}{1-w(0,s)h_2(y)}\int_0^u \frac{d\tau}{-f(\tau)}\Big) - \ln(T-t) \to 0,$$

and

$$\frac{\ln(\frac{1}{1-w(0,s)h_2(y)}\int_0^u \frac{d\tau}{-f(\tau)})}{\ln(T-t)} \to 1, \tag{3.36}$$

uniformly for bounded y, as $t \uparrow T$. It also follows from (3.34) and (3.36) that

$$\frac{g\big(-\ln(\frac{1}{1-w(0,s)h_2(y)}\int_0^u \frac{d\tau}{-f(\tau)})\big)}{g\big(-\ln(T-t)\big)} \to 1, \tag{3.37}$$

Let

$$H = \big(f'(u)(T-t)g(-\ln(T-t))\big)$$
$$\Big(1 - \frac{\int_0^u \frac{d\tau}{-f(\tau)}}{(T-t)(1-w(0,s)h_2(y))}\frac{g\big(-\ln(\frac{1}{1-w(0,s)h_2(y)}\int_0^u \frac{d\tau}{-f(\tau)})\big)}{g\big(-\ln(T-t)\big)}\Big).$$

By Lemma 3.13, and by the formulas (3.35) and (3.37),

$$H \to 0, \tag{3.38}$$

uniformly for bounded y, as $t \uparrow T$.

Finally we get

$$\begin{aligned} &f'(u)(T-t)g(-\ln(T-t)) - \frac{1}{1-w(0,s)h_2(y)} = \\ &H - \frac{1}{1-w(0,s)h_2(y)}\Big(1 - f'(u)\int_0^u \frac{d\tau}{-f(\tau)} g\big(-\ln(\frac{\int_0^u \frac{d\tau}{-f(\tau)}}{1-w(0,s)h_2}\big)\Big), \end{aligned} \tag{3.39}$$

where the last bracket can be written, using (1.4), (3.4) and (3.34), in the form

$$\begin{aligned} &1 - \frac{f'(u)u}{-f(u)}(1+o(1))g\big(-\ln(\frac{u(1+o(1))}{-(1-w(0,s)h_2(y))f(u)})\big) = \\ &1-(1+o(1))\frac{f'(u)u}{-f(u)}g(-\ln(u)). \end{aligned}$$

The right-hand side of this equation converges to zero by (3.32). The claim follows from this, from (3.38) and (3.39). □

Proof of Theorem 3.1 (b). Projecting the equation $w_s = \mathcal{L}w + F$ to the subspace generated by the function h_2, we obtain

$$\sum_k \tilde{a}_k'(s)\langle \tilde{h}_k, h_2\rangle_{L^2_\rho} = \langle \mathcal{L}w, h_2\rangle_{L^2_\rho} + \langle F, h_2\rangle_{L^2_\rho}.$$

Note that $\langle F, h_2\rangle_{L^2_\rho} = 2\int_0^{le^{\frac{1}{2}s}} + \int_{le^{\frac{1}{2}s}}^\infty Fh_2\rho$, where the latter integral is less than $C\exp(-\epsilon e^s)$ by (3.30), and so only the first of these integral is essential in the equation (3.40), as $s \to \infty$. The factor 2 is included in the integrals below, because the solution is symmetric. We can conclude by (3.31) and the orthogonality of the base $\{\tilde{h}_k\}_{k=0}^\infty$, that ($C = 4\sqrt{\pi}$), as $s \to \infty$

$$Ca_2'(s) = 2\int_0^\infty \frac{u_x^2}{f(u)^2}f'(u)h_2(y)\rho(y)dy = 2\int_0^\infty (T-t)f'(u)w_y^2 h_2(y)\rho(y)dy,$$

and

$$Cg(s)a_2'(s) = 2\int_0^\infty (T-t)f'(u)g(-\ln(T-t))w_y^2 h_2(y)\rho(y)dy.$$

Write this in the form

$$Cg(s)a_2' = 2\int_0^\infty (T-t)f'(u)g(-\ln(T-t))(w_y^2 - w(0,s)^2y^2)h_2(y)\rho(y)dy+$$
$$2\int_0^\infty ((T-t)f'(u)g(-\ln(T-t)) - \frac{1}{1-w(0,s)h_2(y)})w(0,s)^2y^2h_2(y)\rho(y)dy+$$
$$2\int_0^\infty \frac{1}{1-w(0,s)h_2(y)}(w(0,s)^2 - a_2(s)^2)y^2h_2(y)\rho(y)dy+$$
$$2\int_0^\infty \frac{a_2(s)^2y^2}{1-w(0,s)h_2(y)}h_2(y)\rho(y)dy = \sum_{j=1}^4 I_j(s). \tag{3.40}$$

Next we show that $I_j(s)\to 0$, as $s\to\infty$ and $j=1,2,3$.

When $j=1$, then by Lemmas 3.3 and 3.13, we can apply the Lebesgue Dominated Convergence Theorem. Writing $w_y^2 - w(0,s)^2y^2 = (w_y + w(0,s)y)$ $(w_y - w(0,s)y)$, we can conclude by Lemmas 3.3, 3.13 and the formula (3.22), that

$$\lim_{s\to\infty} I_1(s) = 0. \tag{3.41}$$

Correspondingly, in the case $j=2$, we obtain from Lemmas 3.3 and 3.14 that

$$\lim_{s\to\infty} I_2(s) = 0. \tag{3.42}$$

When $j=3$, we obtain, by Lemmas 3.3 and 3.12,

$$\lim_{s\to\infty} I_3(s) = 0. \tag{3.43}$$

We can also see that there exist positive constants c_1 and c_2 such that

$$-c_1a_2(s)^2 \le I_4(s) \le -c_2a_2(s)^2, \tag{3.44}$$

for all $s\ge -\ln T$.

By the relations (3.40)- (3.44) and Lemma 3.3 it follows after some calculations that there is $c_3>0$ such that

$$\limsup_{s\to\infty}(g(s)a_2'(s) + c_3a_2(s)^2) \le 0. \tag{3.45}$$

Finally, we conclude that (3.45) implies

$$\lim_{s\to\infty} a_2(s) = 0. \tag{3.46}$$

(1): If a_2 has a non-zero limit a^* ($a^*>0$, because of Lemmas 3.3 and 3.12), then by (3.45) it holds that for every $\varepsilon>0$ there exists a $s_0\ge -\ln T$ and $C>0$ such that $g(s)a_2'\le -C$, as $s\ge s_0$. Integrating this, we obtain

$$a_2(s) - a_2(s_0) \le -C\int_{s_0}^s \frac{d\tau}{g(\tau)} \to -\infty,$$

by (3.33). This is a contradiction to Lemmas 3.3 and 3.12 .

(2): If a_2 does not have a limit, then it follows by Lemmas 3.3 and 3.12 that there exists a sequence $s_j \to \infty$ such that $a_2'(s_j) \geq 0$, and $a_2(s_j) \geq \delta > 0$, which is a contradiction to (3.45).

Theorem 3.1(b) follows from (3.46) and Lemma 3.12. □

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ISBN 951-22-5945-1
ISSN 0784-3143
Institute of Mathematics, Helsinki Univ. of Tech., Espoo, 2002