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RESOLVENT CONDITIONS AND POWERS OF OPERATORS

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Resolvent conditions and powers of operators

Abstract

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0. Introduction

In this paper we discuss powers of bounded operators whose spectrum lies in the closed unit disc. The general theme is to relate growth conditions of the resolvent near the unit disc to bounds for the powers and their differences.

Much of my present interest in these questions originated from a question of J.Zemanek who asked whether there are quasinilpotent operators Q such that $A = 1 + Q$ would satisfy the *Ritt resolvent condition*, i.e for $|\lambda| > 1$

$$||(\lambda - A)^{-1}|| \leq \frac{C}{|\lambda - 1|}.$$

This condition has now a characterization (see Theorem 10 below) but we still do not know the answer to the original question. Other properties which can be characterized or estimated quantitatively include:

The resolvent is uniformly bounded outside unit disc (see Theorem 1).

Operators which are power bounded even after a small overrelaxation (see Theorem 8).

Power boundedness for operators which have meromorphic resolvent in a neighborhood of the unit circle (see Theorem 12).

We should also mention that Strikwerda and Wade [SW] have shown that the *Kreiss resolvent condition*

$$||(\lambda - A)^{-1}|| \leq \frac{C}{|\lambda| - 1}$$

is equivalent with boundedness of the second Cesaro sums of the powers of $e^{i\theta}A$.

In addition to powers as such we also look at their differences, in particular the behavior of $A^n(A-1)$. A result of Katznelson and Tzafriri [K] characterizes those power bounded operators for which the differences tend to zero. We give some examples which e.g. show that the conclusion does not hold if power boundedness is replaced by the Kreiss resolvent condition. Also, we look at a condition

$$||e^{zA}|| \leq Ce^{|z|}$$

which is weaker than power boundedness but stronger than the Kreiss condition (see Theorem 7). Here again it is natural to ask whether these conditions would imply much stronger conclusions in the case $A = 1 + Q$. We are able to show only that the bounds can be replaced by the corresponding "little oh"-versions if the spectrum touches the unit disc in a set of zero measure. We give a list of results first and postpone the proofs to the end.

Much of this research was done while the author was visiting at the Mittag-Leffer Institute during the winter 1997-1998.

1. Results

We study the growth and decay of powers of operators in Banach spaces. So, let X be a complex Banach space, A a bounded operator in X such that the spectrum is in the closed unit disc

$$\sigma(A) \subset \overline{\mathbb{D}}. \tag{1.1}$$

We shall partly specialize to the case where $\sigma(A) = \{1\}$ and then we write $A = 1+Q$ with $\sigma(Q) = \{0\}$. The resolvent is denoted by $(\lambda - A)^{-1}$ and is analytic outside the unit disc. If we put

$$M(r) := \sup_{|\lambda| \geq r} \|(\lambda - A)^{-1}\| \tag{1.2}$$

then (1.1) can be written equivalently as

$$M(r) < \infty \text{ for } r > 1 \tag{1.3}$$

Since for $n \geq 0$

$$A^n = \frac{1}{2\pi i} \int_{|\lambda|=r} \lambda^n (\lambda - A)^{-1} d\lambda \tag{1.4}$$

this immediately gives for $r > 1$

$$\|A^n\| \leq M(r) r^{n+1} \tag{1.5}$$

and in particular as $n \to \infty$

$$\|A^n\| \leq e^{o(n)} \tag{1.6}$$

Example 1. Operators of the form $A = 1 + Q$ with Q quasinilpotent can have fastly growing powers. In fact, let ω be a positive real and choose $\alpha_j = (\frac{1}{j})^{1/\omega}$ and put Q for the weighted backward shift $Qe_1 = 0$ while $Qe_{j+1} = \alpha_j e_j$. Then Q is quasinilpotent and its resolvent grows like $e^{\tau_1(\omega)/r^\omega}$, that is with order ω and with a posivitive type $\tau_1(\omega)$. The growth of the resolvent and the growth of the Taylor coefficients are related and one obtains likewise that $\|(1+Q)^n\|$ grows like $e^{\tau_2(\omega) n^{\frac{\omega}{\omega+1}}}$.

We are mainly interested in the situation where $M(r) \to \infty$ as r tends to 1. However, for the sake of completenes, we formulate also a result for the case where $M(1) < \infty$.

Theorem 1. *If $M(1) < \infty$, then*

$$M(1) \leq \sum_0^\infty \|A^n\| \leq 1 + 4[M(1) - 1]M(1) \tag{1.7}$$

Remark 1.

This result was given in [Ne5] where the constant 6 appeared in place of 4, in (1.7). For example, taking a nilpotent two dimensional matrix with a small norm we see that the constant must be larger than 2. For large values of $M(1)$ the behavior is necessarily quadratic in $M(1)$, see [Ne5].

From now on we shall assume $M(1) = \infty$. In Example 1 we saw that the growth of powers can be fast even in the case were the spectrum touches the unit circle only at one point. We shall first pose a restriction of different nature. We assume that the resolvent is of *bounded characteristics* outside the unit disc. To that end put

$$m(r) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ ||(re^{i\phi} - A)^{-1}|| d\phi. \tag{1.8}$$

This tool would suffice if we would only look at resolvents being analytic outside the unit disc and have an essential singularity somewhere on the unit circle. In general, however, one has to count also the number of poles. Assume that the resolvent is meromorphic for $|\lambda| > R$ and let b be a pole, meaning that there is a smallest positive integer μ, the multiplicity of b, such that $(\lambda - b)^{\mu}(\lambda - A)^{-1}$ is analytic in a neighborhood of b. Number the poles so that their absolute values are nonincreasing, repeating each pole according to the multiplicity, as long as they stay outside the disc of radius R. Then we set for $r > R$

$$N(r) := \sum_j \log^+ \frac{|b_j|}{r} \tag{1.9}$$

Finally we denote

$$T(r) := m(r) + N(r). \tag{1.10}$$

Now one says that the resolvent is of *bounded characteristics* outside the disc of radius R if

$$T(R) := \limsup_{r \to R+} T(r) < \infty. \tag{1.11}$$

One knows that T is nondecreasing and convex in the the variable $\log \frac{1}{r}$, see e.g. [Ne4]. Notice that since we assume that the spectrum is in the unit disc $T(r) = m(r)$ for $r \geq 1$.

Theorem 2. *Suppose (1.1) holds and* $T(1) < \infty$. *Then for all* $n \geq 0$ *we have*

$$||A^n|| \leq e^{T(1)} e^{\sqrt{8T(1)(n+1)}} \tag{1.12}$$

Remark 2. The best constants are not known. The theorem is sharp in the sense that for the operators in Example 1 we have $T(1) < \infty$ for $\omega < 1$ while $T(1) = \infty$ for $\omega > 1$.

Example 2. Let V denote the integration in $L_2[0,1]$

$$Vf(t) = \int_0^t f(s)ds$$

and put $A = 1 + V$. Then A^{-1} is a contraction and it follows from Phragmen-Lindelöf principle that for all $\epsilon > 0$ there does not exist a constant C_ϵ so that for all n

$$\|A^n\| \leq C_\epsilon e^{\epsilon\sqrt{n}},$$

see [A]. On the other hand, one sees easily that

$$\|A^n\| \leq e^{2\sqrt{n}}$$

holds. Further, $\sigma(1+V) = \{1\}$, $m(1) < \infty$ but V is not in the trace class as the singular values decay like $1/j$.

The next result gives a sufficient condition for $1+K$ to satisfy $T(1) < \infty$. Notice that the result does not depend on the location of the spectrum.

Theorem 3. *Suppose K is a compact operator in a Hilbert space such that the singular values $\sigma_j(K)$ satisfy*

$$C_*(K) := \sum_1^\infty \frac{\sigma_j(K)}{2} \log^+ \frac{2}{\sigma_j(K)} < \infty. \tag{1.13}$$

Then for $A = 1 + K$ we have

$$T(1) \leq C_*(K) + \|K\|_1.$$

where $\|K\|_1 = \sum_{j=1}^\infty \sigma_j(K)$.

We shall now pose a natural growth condition for the resolvent near the unit circle. If A satisfies $\|A^n\| \leq C$ for $n \geq 0$ then the expansion

$$(\lambda - A)^{-1} = \sum A^n \lambda^{-1-n}$$

implies the *Kreiss resolvent condition*

$$\|(\lambda - A)^{-1}\| \leq \frac{C}{|\lambda| - 1} \text{ for } |\lambda| > 1$$

which we shall write here as

$$M(r) \leq \frac{C}{r-1} \text{ for } r > 1. \tag{1.14}$$

Substituting $r := 1 + 1/n$ into (1.5) gives

$$\|A^n\| \leq Ce(n+1) \tag{1.15}$$

which is optimal in the following sense: no smaller constant than Ce is possible and the dependency on n can be linear. In fact, the former can be seen by looking at truncated shift operators multiplied by a very large constant [Le]. Recent sharp forms of this are proved in [Sp]. The dependency on n is central to our theme and we give an example by Shields [Sh] in detail.

Example 4, [Sh]. Let X denote the space of analytic functions in the open unit disc such that f' has boundary values in the Hardy space H^1, equipped with the norm

$$||f|| := |f|_\infty + |f'|_1 = \sup_{|z|\le 1} |f(z)| + \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(e^{i\varphi})| d\varphi.$$

If M_z denotes the multiplication operator with the variable z then $||M_z^n|| = n+1$ while the Kreiss condition holds (e.g. $C = 3/2$ will do). Also, the following holds (see the proof of Example 5) for $t > 0$

$$(1 + \frac{1}{2}\sqrt{t})e^t \le ||e^{tM_z}|| \le (1 + 2\sqrt{t})e^t.$$

Also here the growth is as fast as it can be..

Theorem 4. *If the Kreiss condition (1.14) holds, then for all* z

$$||e^{zA}|| \le C_1\sqrt{1+|z|}e^{|z|} \tag{1.16}$$

with $C_1 = 2C$. *If (1.16) holds, then for all* $n \ge 0$

$$||A^n|| \le C_1\sqrt{2\pi}\ (n+1). \tag{1.17}$$

In the previous example the spectrum of M_z equals the closed unit disc. If we assume that the spectrum touches the unit circle only in a set of measure zero then a sharpening of (1.15) can be given. Let us denote the usual measure on the unit circle by $meas$, normalized so that $meas(\partial\mathbb{D}) = 1$. Notice that if $T(1) < \infty$ then always $meas(\sigma(A) \cap \partial\mathbb{D}) = 0$.

Theorem 5. *Assume that the Kreiss condition (1.14) holds and that*

$$meas(\sigma(A) \cap \partial\mathbb{D}) = 0.$$

Then as $n \to \infty$

$$||A^n|| = o(n). \tag{1.18}$$

This result has a converse.

Theorem 6. *There exists a Banach space with the following property. Let* E *be a closed set of the unit circle. Then there exists an operator* A *satisfying the Kreiss condition,*

$$\sigma(A) \cap \partial\mathbb{D} = E$$

and such that

$$||A^n|| \ge 1 + \frac{n}{2}\ meas(E). \tag{1.19}$$

Example 5.

We set $B := \frac{1}{2}(1 + M_z)$ where M_z is the multiplication operator in the space X of Example 4. Then the Kreiss condition (1.14) holds and

$$\sigma(B) = \{\lambda | \, |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$$

so that in particular

$$\sigma(B) \cap \partial\mathbb{D} = \{1\},$$

$$T(1) < \infty,$$

there exists C_1 for $t > 0$ we have

$$(1 + \frac{1}{C_1}\sqrt{t})e^t \leq \|e^{tB}\| \leq (1 + C_1\sqrt{t})e^t$$

there exists C_2 such that

$$\|(\lambda - B)^{-1}\| \leq \frac{C_2}{|\lambda - 1|^2} \text{ for } 1 < |\lambda| < 2,$$

there exists C_3 such that for $n \geq 0$

$$\frac{1}{C_3}\sqrt{n+1} \leq \|B^n\| \leq C_3\sqrt{n+1},$$

there exists C_4 such that for $n \geq 0$

$$\frac{1}{C_4} \leq \|B^n(B - 1)\| \leq C_4.$$

Remark 3. We recall ([Ne5] Proposition 1.1, or Theorem 6 below) that the iterated resolvent condition

$$\|(\lambda - A)^{-n}\| \leq \frac{C}{(|\lambda| - 1)^n} \quad \text{for } n \geq 0, \ |\lambda| > 1 \tag{1.20}$$

is equivalent with the condition

$$\|e^{zA}\| \leq Ce^{|z|}. \tag{1.21}$$

So, in particular, the operator B in Example 5 does not satisfy the iterated resolvent condition.

Notice further that by the well known theorem by Katznelson and Tzafriri [K] power boundedness and

$$\sigma(A) \cap \partial\mathbb{D} \subset \{1\} \tag{1.22}$$

together imply

$$A^n(A - 1) \to 0. \tag{1.23}$$

Thus we see that power boundedness cannot be replaced in this result by the weaker Kreiss resolvent condition. Whether power boundedness can be replaced by the iterated resolvent condition is open.

Theorem 7. *The following holds:*
If for $n \geq 0$

$$\|A^n\| \leq C \tag{1.24}$$

then both (1.20) and (1.21) follow. These are equivalent and both imply (1.14) and

$$\|A^n\| \leq C\sqrt{2\pi(n+1)}. \tag{1.25}$$

If the iterated resolvent condition (1.20) (and (1.21)) holds and additionally

$$meas(\sigma(A) \cap \partial\mathbb{D}) = 0$$

then as $n \to \infty$

$$\|A^n\| = o(\sqrt{n}). \tag{1.26}$$

Example 6. We shall now look at the effect of under- (and over-)relaxation. To that end we start with a simple example. Consider e.g. continuous 2π-periodic functions with maximum norm and let the operator be multiplication by $e^{i\theta}$ composed with underrelaxation:

$$(A_\omega)f(\theta) := (\omega e^{i\theta} + 1 - \omega)f(\theta)$$

where $0 < \omega < 1$.
Then

$$\|e^{te^{i\varphi}A_\omega}\| \leq e^{t(1-\frac{2(1-\omega)\varphi^2}{\pi^2})} \tag{1.27}$$

for $|\varphi| \leq \pi$. On the other hand, there is a constant $c > 0$ such that for all $n \geq 1$

$$\|A_\omega^n(A_\omega - 1)\| \geq c\sqrt{\frac{\omega}{(1-\omega)n}} \tag{1.28}$$

Theorem 4.5.3 in [Ne1] says that if contractions are underrelaxed then the differences decay (at least) like $1/\sqrt{n}$. The next theorem says that the same happens also for operators which need not to be power bounded as long as they satisfy the iterated resolvent condition. Example 5 shows in particular that there are operators which satisfy the Kreiss condition but which do not satisfy the iterated resolvent condition even after underrelaxation and the differences do not decay.

Our next result characterizes the operators which are power bounded even after a small overrelaxation.

Theorem 8. *The following are equivalent:*

$$\|(re^{i\varphi} - A)^{-n}\| \leq \frac{C}{(r + c\varphi^2 - 1)^n} \tag{1.29}$$

$$\|e^{te^{i\varphi}A}\| \leq Ce^{t(1-c\varphi^2)}. \tag{1.30}$$

If either of them holds then for $0 \le \epsilon \le \frac{c}{1-2c}$ *we have for* $A_{1+\epsilon} := (1+\epsilon)A - \epsilon$

$$\|A_{1+\epsilon}^n\| \le C_1 \tag{1.31}$$

where $C_1 = C\sqrt{\frac{2}{c}}$.

Reversely, if (1.31) holds then (1.29) and (1.30) hold with $C = C_1$ *and* $c = \frac{\epsilon}{(1+\epsilon)\pi^2}$.

Further, if (1.29) holds then

$$\|A^n(A-1)\| \le \frac{C_2}{\sqrt{n+1}}. \tag{1.32}$$

where $C_2 = \frac{C}{c\sqrt{\pi}}$.

Corollary 1. *If* A *satisfies the iterated resolvent condition (1.20), then the under-relaxed operators* A_ω *are power bounded, and the following estimates hold:*

$$\|A_\omega^n\| \le \frac{C\pi}{\sqrt{1-\omega}} \tag{1.33}$$

and

$$\|e^{te^{i\varphi}A_\omega}\| \le Ce^{t(1-\frac{2(1-\omega)\varphi^2}{\pi^2})} \tag{1.34}$$

where $0 \le \omega < 1$ *and* $|\varphi| \le \pi$.

In fact, since (1.20) and (1.21) are equivalent we have

$$\|e^{zA}\| \le Ce^{|z|}.$$

Thus for $A_\omega = \omega A + 1 - \omega$

$$\|e^{te^{i\varphi}A_\omega}\| \le Ce^{t\omega}e^{(1-\omega)t\cos\varphi} \le Ce^{t(1-\frac{2(1-\omega)\varphi^2}{\pi^2})}.$$

The estimate for powers of A_ω now follows from the previous theorem.

The next result can be compared with with Example 5 by setting $\alpha = 1$.

Theorem 9. *Assume the Kreiss condition (1.14) holds and that there exists* $\alpha \ge 0$ *and* C_1 *such that for* $1 < |\lambda| < 2$

$$\|(\lambda - A)^{-1}\| \le \frac{C_1}{|\lambda - 1|^{1+\alpha}}, \tag{1.35}$$

then for every integer $k \ge 0$ *there exists* M_k *such that for all* $n \ge 0$

$$\|A^n(A-1)^k\| \le \frac{M_k}{(n+1)^\beta} \tag{1.36}$$

where $\beta = \frac{k-\alpha}{1+\alpha}$.

Here the case $\alpha = 0$ corresponds to the *Ritt condition*

$$\|(\lambda - 1)(\lambda - A)^{-1}\| \le C \text{ for } |\lambda| > 1. \tag{1.37}$$

This condition is formally weaker as the sectorial condition studied in [Ne1] and in [Ne5]. However, it is contained in the proof of the previous theorem that the condition easily extends into a sectorial set. This has been shown also in [Ly] and [Na]. Finally, we mention that [B] contains an integration argument where the Ritt condition implies directly the power boundedness. Ritt originally showed that the condition implies growth at most of the form $o(n)$, which was improved in [T] to $O(\log(n))$. Theorem 2.1 in [Ne5] gives 4 different characterizations for (1.36) to hold with $k = 0$ and 1. We formulate here still one (already published in [Na]) as follows.

Theorem 10. *The following are equivalent:*

(i) There exists M_0 and M_1 such that for all $n \geq 0$

$$||A^n|| \leq M_0 \tag{1.38}$$

and

$$||A^n(A-1)|| \leq \frac{M_1}{n+1}. \tag{1.39}$$

(ii) There exists C such that the Ritt condition (1.37) holds.

The we can still strenghten (1.37) by assuming that the corresponding iterated version holds. This then already implies that $A = 1$. In fact, already a somewhat weaker assumption impilies this.

Theorem 11. *Assume that the Ritt condition (1.37) holds and that for some $\lambda \leq -1$ we have for all $n \geq 0$*

$$||(1 - \frac{1}{\lambda - 1}(A-1))^{-n}|| \leq C_1. \tag{1.40}$$

Then $A = 1$.

We recall that Spijker has proved that if A is a matrix in a $d-$dimensional space then Kreiss condition implies

$$||A^n|| \leq Ced. \tag{1.41}$$

In a $d-$dimensional space the resolvent is meromorphic in the whole plane. In the general case we can assume that the resolvent is meromorphic outside a disc which is strictly smaller than the unit disc. Then power boundedness can be estimated in terms of $T(r)$ and the constant C in the Kreiss condition.

Theorem 12. *For every $0 < \theta < 1$ there exists a constant $C_1(\theta)$ such that the following holds: If $T(\theta) < \infty$ then the Kreiss condition (1.14) implies for $n \geq 0$*

$$||A^n|| \leq C_1(\theta)T(\theta)C. \tag{1.42}$$

This is just a recombination of results in [Ne2]. Here is a corollary for operators in Hilbert spaces.

Theorem 13. *For every $0 \leq \eta < 1$ there exists a constant $c(\eta)$ such that the following holds: If A is an operator in a separable Hilbert space satisfying the Kreiss condition (1.14) and such that it can be decomposed as $A = B + K$ where $||B|| \leq \eta$ and $||K||_1 < \infty$. Then we have for $n \geq 0$*

$$||A^n|| \leq c(\eta)[||K||_1 + 1]C. \tag{1.43}$$

An earlier version of Theorem 12 was given in [Ne5], with a different growth function to measure the sice of the resolvent as a meromorphic function. The natural growth function here is, however, $T(r)$, and we point out that also other results in section 4 [Ne5] have counterparts in this terminology. For example, if we only assume (1.39), then a resolvent condition follows which implies (1.39) back provided $T(\theta) < \infty$ with some $\theta < 1$.

We shall end with a remark on the small values of $T(1)$. For the identity operator we have $T(1) = \gamma$, where

$$\gamma := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |e^{i\theta} - 1| d\theta = 0.323...$$

If $T(1) < \gamma$, then the spectral radius of the operator is necessarily smaller than 1 and we can bound the size of $M(1)$ in terms of $T(1)$.

Theorem 14. *If $T(1) < \gamma$, and if ξ is the solution of*

$$\xi = \frac{1}{3(\gamma - T(1))}(1 + \log(1 + \xi))$$

then

$$M(1) \leq \xi.$$

Example 7. We can also look at the continuity from "above". Operators of the form $1+\epsilon Q$ where $\epsilon > 0$, Q is as in Example 1 with $\omega < 1$, are examples of operators satisfying $\gamma < T(1) < \infty$ and having fastly growing powers. From writing

$$(e^{i\theta} - 1 - \epsilon Q)^{-1} = \frac{1}{e^{i\theta} - 1}(1 - \frac{\epsilon}{e^{i\theta} - 1}Q)^{-1}$$

we see that for a fixed Q and small ϵ we have $T(1) < \gamma + C\epsilon$.

2. Proofs

Proof of Theorem 1.

For short, let $M := M(1)$. The idea of the proof is simple. Knowing the value of M allows us to use estimate (1.5)

$$||A^n|| \leq M(r)r^{n+1}$$

with $r < 1$ such that $(1 - r)M < 1$. In fact, we obtain from the identity

$$(\lambda - A)^{-1} = (\lambda_0 - A)^{-1}(1 - (\lambda - \lambda_0)(\lambda_0 - A)^{-1})^{-1}$$

that

$$M(r) \leq \frac{M}{1 - (1 - r)M}.$$

We choose $r = 1 - \frac{1}{2M-1}$ which gives

$$M(r) \leq \frac{(2M - 1)M}{M - 1}$$

and so

$$\sum_0^\infty ||A^n|| \leq 1 + r\sum_1^\infty M(r)r^n = 1 + r^2\frac{M(r)}{1 - r} \leq 1 + 4(M - 1)M.$$

Proof of Theorem 2.

Since the resolvent is analytic outside the unit circle and bounded at infinity we can bound the maximum norm outside a circle by the logarithmic average along a slightly smaller circle. In fact, since the integrand in m is subharmonic one gets using the Poisson kernel for $1 < \theta < r$

$$\log^+ M(r) \leq \frac{\theta+1}{\theta-1} m(r/\theta)$$

(a formulation for operator valued functions is given in [Ne4]) which by assumption on the limit $\theta \to r$ gives

$$\log^+ M(r) \leq \frac{r+1}{r-1} m(1).$$

We utilize this by substituting $r := 1 + \sqrt{\frac{2\ m(1)}{n+1}}$ into (1.5) so that

$$\|A^n\| \leq M(r) r^{n+1} \leq r^{n+1} e^{\frac{r+1}{r-1} m(1)} \leq e^{m(1)}\ e^{\sqrt{8m(1)(n+1)}}.$$

Proof of Theorem 3.

Here we assume that $A = 1 + K$ where K is a compact operator in a Hilbert space such that

(2.1) $$\|K\|_1 = \sum \sigma_j < \infty$$

and

(1.13) $$C_*(K) := \sum (\sigma_j/2) \log^+(\frac{2}{\sigma_j}) < \infty.$$

Here $\sigma_j = \sigma_j(K)$ denote the singular values of the operator (ordered nonincreasingly). (Finite dimensional cases are included with trivial modifications.) Notice that from the compactness of K it follows that $\sigma_j \to 0$ and so (1.13) implies (2.1).

In [Ne4] two characteristic functions, T_∞ and T_1 for operator valued meromorphic functions where discussed. The functions were assumed to be meromorphic in a disc $|z| < R$ and normalized as to be the identity at origin.

Here we want to estimate $T(1)$ of the resolvent $(\lambda - 1 - K)^{-1}$. In order to be able to refer to the results directly we change the variable $\lambda = 1/z$ and write

$$(\lambda - 1 - K)^{-1} = \frac{z}{1-z}(1 - \frac{z}{1-z}K)^{-1}.$$

At $z = 1$ we have $T(1) = T_\infty(1)$. Clearly, $(1 - \frac{z}{1-z}K)^{-1}$ is meromorphic in the open disc. Thus we have

$$T(1) \leq \sup_{t<1} T_\infty(t, (1 - \frac{z}{1-z}K)^{-1}).$$

On the other hand, since K is in trace class, the following estimate holds:

$$T_\infty(t,(1-\frac{z}{1-z}K)^{-1}) \le T_1(t,1-\frac{z}{1-z}K),$$

see [Ne4].

Since $1-\frac{z}{1-z}K$ is analytic in the open unit disc, T_1 is here simply

$$T_1(t,1-\frac{z}{1-z}K) = \sum_{j=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \sigma_j d\varphi$$

where now $\sigma_j = \sigma_j(1-\frac{te^{i\varphi}}{1-te^{i\varphi}}K)$ are the singular values.

Estimate $\sigma_j(1-\frac{z}{1-z}K) \le 1+\frac{|z|}{|1-z|}\sigma_j(K)$ follows from the approximation propoerty, from the fact that the singular values are obtained as distances to finite rank operators. Thus, we have to estimate integrals of the following form

$$F(\sigma) := \frac{1}{2\pi}\int \log(1+\frac{\sigma}{|1-e^{i\varphi}|})d\varphi.$$

Then $T_1(1,1-\frac{z}{1-z}K) \le \sum_j F(\sigma_j(K))$.

Estimate here $|1-e^{i\varphi}| \ge \frac{2}{\pi}\varphi$ and substitute $\varphi = \pi\sigma/2u$. Then for $u \le 1$ estimate $\log(1+u) \le u$ while for $u > 1$ estimate $\log(1+u) \le \sqrt{u}$. This gives

$$F(\sigma) \le \frac{\sigma}{2}\log^+ \frac{2}{\sigma} + \sigma,$$

and the bound

$$T(1) \le C_*(K) + ||K||_1$$

follows.

Proof of Theorem 4.

To prove (1.16) we may put $z = t$ as the Kreiss condition holds for A if and only if it holds for $e^{i\theta}A$. Write

$$e^{tA} = \frac{r}{2\pi}\int e^{tre^{i\varphi}}(re^{i\varphi}-A)^{-1}e^{i\varphi}d\varphi \tag{2.2}$$

to obtain

$$||e^{tA}|| \le \frac{Cr}{r-1}\frac{1}{2\pi}\int |e^{tr\cos\varphi}|d\varphi. \tag{2.3}$$

Substituting here $r = 1+1/t$ yields the bound.

Suppose now that (1.16) holds.

We use the following representation for powers of A

$$A^n = \frac{n!}{2\pi i}\int z^{-1-n}e^{zA}dz \tag{2.4}$$

where the integration is around the origin. Choose $|z| = n$. Then we obtain

$$||A^n|| \le 2C\sqrt{n+1}e^n\frac{n!}{n^n}$$

which implies (1.17).

Proof of Theorem 5.

Choose $\epsilon > 0$. We have to show that

$$\limsup \|A^n\|/n \leq \epsilon. \tag{2.5}$$

The set $E := \sigma(A) \cap \partial\mathbb{D})$ is compact and of zero measure. Choose an open cover $\{U_j\}$ of E on the circle such that

$$\sum_j meas(U_j) < \frac{\epsilon}{Ce}$$

where C is the constant in the Kreiss condition, and each U_j is an arc along the circle. By the compactness of E the cover can be assumed to be finite, say $j = 1, 2, \ldots, N$ and nonoverlapping. Consider now φ such that $e^{i\varphi} \notin \cup_1^N U_j$ and such that $re^{i\varphi} \in \sigma(A)$. From the compactness of $\sigma(A)$ it follows that there exists a $\delta > 0$ such that for all such φ we have $r \leq 1 - \delta$. We use the Cauchy integral

$$A^n = \frac{1}{2\pi i} \int_\Gamma \lambda^n (\lambda - A)^{-1} d\lambda \tag{1.4}$$

where Γ consists of a finite number of pieces as follows:
when $e^{i\varphi} \in U_j$ we choose $\lambda := (1 + \frac{1}{n+1})e^{i\varphi}$
when $e^{i\varphi} \notin \bar{U}_j$ we choose $\lambda := (1 - \delta)e^{i\varphi}$
while inbetween when $e^{i\varphi} \in \partial U_j$ we keep φ fixed and have $\lambda := te^{i\varphi}$.
The first choice corresponds to, using the Kreiss condition

$$Ce(n+1) \sum_j meas(U_j) \leq \epsilon(n+1),$$

the second terms all decay with speed $(1 - \delta)^n$ while the third terms decay with speed $1/(n+1)$.

This completes the proof.

Proof of Theorem 6.

The proof is based on Example 5. Let $E \subset \partial\mathbb{D}$ be a closed set of positive measure. We define a Banach space X_E as follows. For every $e^{i\theta} \in E$ let f_θ be an analytic function in the unit disc with boundary values in H^1. Denote by f the set $\{f_\theta\}_{e^{i\theta} \in E}$. We set

$$\|f\| := \sup_\varphi \sup_\theta |f_\theta(e^{i\varphi})| + \frac{1}{2\pi} \int_\pi^\pi \sup_\theta |f'_\theta(e^{i\varphi})| d\varphi. \tag{2.6}$$

and define Y_E as the closure of those with finite norm.

Next, our operator A shall be a diagonal multiplication operator as follows. The component f_θ shall be multiplied by $(e^{i\theta} + z)/2$ where z indicates the variable. Thus

$$(Af)_\theta(z) = \frac{e^{i\theta} + z}{2} f_\theta(z)$$

and in particular

$$(A^n f)_\theta(e^{i\theta}) = e^{in\theta} f_\theta(e^{i\theta})$$
$$(A^n f)'_\theta(e^{i\theta}) = \frac{n}{2} e^{i(n-1)\theta} f_\theta(e^{i\theta}) + e^{in\theta} f'_\theta(e^{i\theta}).$$

Applied to the constant vector f with $f_\theta(z) = 1$ we obtain

$$||A^n f|| \geq 1 + \frac{n}{2} meas(E).$$

We still have to check that the Kreiss condition holds and that $\sigma(A) \cap \partial\mathbb{D} = E$.
Let $|\lambda| = r > 1$. Since

$$((\lambda - A)^{-1} f)_\theta(z) = (\lambda - \frac{e^{i\theta} + z}{2})^{-1} f_\theta(z)$$

we have

$$\sup_{\varphi,\theta} |((\lambda - A)^{-1} f)_\theta(e^{i\varphi})| \leq \frac{1}{r-1} \sup_{\varphi,\theta} |f_\theta(e^{i\varphi})|.$$

Similarly,

$$((\lambda - \frac{e^{i\theta} + z}{2})^{-1} f_\theta)'(z) = \frac{1}{2}(\lambda - \frac{e^{i\theta} + z}{2})^{-2} f_\theta(z) + (\lambda - \frac{e^{i\theta} + z}{2})^{-1} f'_\theta(z)$$

gives

$$\frac{1}{2\pi} \int \sup_\theta |((\lambda - \frac{e^{i\theta} + e^{i\varphi}}{2})^{-1} f_\theta)'(e^{i\varphi})| d\varphi$$
$$\leq \frac{1}{2} \frac{1}{2\pi} \int \sup_\theta |(\lambda - \frac{e^{i\theta} + e^{i\varphi}}{2})^{-2}| d\varphi \sup_{\varphi,\theta} |f_\theta(e^{i\varphi})| + \frac{1}{2\pi} \int \sup_\theta |f_\theta(e^{i\varphi})|.$$

Together these imply

$$||(\lambda - A)^{-1} f|| \leq \frac{1}{r-1} ||f|| + c(r) \sup_{\varphi,\theta} |f_\theta(e^{i\varphi})|$$

where

$$c(r) = \frac{1}{2} \frac{1}{2\pi} \int \sup_\theta |(\lambda - \frac{e^{i\theta} + e^{i\varphi}}{2})^{-2}| d\varphi \leq \frac{1}{2} \frac{1}{r-1}.$$

Let us now consider the spectrum of A. It is actually the union of discs of the form $|\lambda - e^{i\theta}/2| \leq 1/2$ but all we need here is that its intersection with the unit circle equals E. For simplicity, let us assume that $1 \in E$, other values are similar. Consider the operator M_z in X, as in Example 4. Its spectrum is the closed unit disc. In Example 5 we consider $B = \frac{1}{2}(1 + M_z)$. By the spectral mapping theorem its spectrum is the disc $|\lambda - 1/2| \leq 1/2$ and in particular 1 is in the spectrum. But if we restrict A to the one-dimensional subspace coresponding to $\theta = 0$ we see that it operates like B and 1 is also in the spectrum of the restriction of A. Finally, as the resolvent is analytic outside the disc, 1 is a boundary point and the boundary of of the spectrum of restricted operators always belongs to the spectrum of the full operator.

Take now $e^{i\psi} \notin E$. We have $|e^{i\psi} - \frac{e^{i\theta}+e^{i\varphi}}{2}| \geq \beta|\theta - \psi|^2$, for some $\beta > 0$ not depending on φ. This implies

$$||(e^{i\psi} - A)^{-1}|| \leq \frac{C/\beta}{dist(e^{i\psi}, E)^4}$$

and in particular $e^{i\psi} \notin E$ is a regular value for A.

Above we have formulated for every closed E a Banach space Y_E. In the theorem it is said that only Y space suffisses. This is easily obtained by defining $Y := Y_{\partial\mathbb{D}}$ and treating Y_E as a subspace of Y.

Details to Example 5.

Let M_z and X be as in Example 4 and set $B = \frac{1}{2}(1 + M_z)$. If $b(z) = \frac{1}{2}(1+z)$, then in X we have

$$||B^n|| = 1 + \sqrt{\frac{n}{2\pi}}(1 + o(1)).$$

In fact,

$$||B^n|| = 1 + |(b^n)'|_1$$

as $B^n 1(z) = b^n(z)$ and

$$\begin{aligned} ||B^n f|| &\leq |b^n f|_\infty + |(b^n)' f|_1 + |b^n f'|_1 \\ &\leq ||f|| + |(b^n)'|_1 |f|_\infty \end{aligned}$$

But $(b^n)' = \frac{n}{2} b^{n-1}$ and $|b(e^{i\varphi})| = |\cos\frac{\varphi}{2}|$ so that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} |(b^n)'| d\varphi &= \frac{n}{2\pi}\int_0^{\pi} (\cos\frac{\varphi}{2})^{n-1} d\varphi \\ &= \frac{\sqrt{n-1} + 1/\sqrt{n-1}}{2\pi}\int_0^{\pi\sqrt{n-1}} (\cos\frac{t}{2\sqrt{n-1}})^{n-1} dt \end{aligned}$$

from which the claim follows as the integral tends to

$$\int_0^\infty e^{-t^2/8} dt = \sqrt{2\pi}.$$

Likewise, to obtain

$$\frac{1}{C} \leq ||B^n(B-1)|| \leq C$$

the key term to estimate is $nb^{n-1}(b-1)$.

Consider now the iterated resolvent condition. We first show that the multiplication operator M_z does not satisfy it. This is in fact clear already from the linear growth of the powers as the iterated condition allows at most $\sqrt{n}$ growth. However, the estimate for the exponential function e^{tM_z} has interest in its own right and then the corresponding result for B follows.

It is clear from the previous discussion that to estimate the operator norm of e^{tM_z} we need to compute the norm of e^{tz} as a function in X.

Thus,

$$||e^{tM_z}|| = |e^{tz}|_\infty + |te^{tz}|_1$$

which gives easily the following bound for $t > 0$:

$$(1 + \frac{1}{2}\sqrt{t})e^t \leq ||e^{tM_z}|| \leq (1 + 2\sqrt{t})e^t.$$

Now we obtain from this the corresponding result for B as

$$e^{tB} = e^{\frac{t}{2}M_z} e^{\frac{t}{2}}.$$

Proof of Theorem 7.

This result is contained in Proposition 1.1 in [Ne5], except the case of peripheral spectrum being of zero measure.

We use the following representation for powers of A

$$A^n = \frac{n!}{2\pi i}\int z^{-1-n}e^{zA}dz \tag{2.4}$$

where the integration is around the origin. We assume that for all z

$$\|e^{zA}\| \le Ce^{|z|} \tag{1.21}$$

holds. Our aim is to show that this implies, together with $meas(\sigma(A)\cap\partial\mathbb{D}) = 0$ that for

$$\|A^n\| = o(\sqrt{n}) \tag{1.26}$$

Fix an $\epsilon > 0$. As in the proof of Theorem 4 we may assume that disjoint open arcs U_j are given on the unit circle such that $meas(\cup_1^N U_j) \le \epsilon$ and

$$\sigma(A)\cap\partial\mathbb{D} \subset \cup_1^N U_j \subset \partial\mathbb{D}. \tag{2.5}$$

The integration path consists of several parts. First, let Γ_A denote the following arcs: $z = ne^{i\varphi}$ where $e^{i\varphi} \in \cup_1^N U_j$. Then these contribute to A^n using (2.4) and (1.21)

$$\|\frac{n!}{2\pi i}\int_{\Gamma_A} z^{-1-n}e^{zA}dz\| \le C(e/n)^n n!\epsilon \le C\epsilon\sqrt{2\pi(n+1)}. \tag{2.6}$$

Let now θ stand for an angle such that $e^{-i\theta} \in K := \partial\mathbb{D} - \cup_1^N U_j$. Clearly K is a compact set. We need to estimate $e^{ne^{i\theta}A}$. To that end we fix any such θ_0 and write

$$e^{ne^{i\theta_0}A} = \frac{1}{2\pi i}\int_{\gamma_0} e^{ne^{i\theta_0}\lambda}(\lambda - A)^{-1}d\lambda. \tag{2.7}$$

Here γ_0 is a contour around the spectrum, such that $\Re\{e^{i\theta_0}\lambda\} \le 1 - \epsilon_0$ for some positive ϵ_0 and for all $\lambda \in \gamma_0$. This is possible by the spectral mapping theorem. We obtain

$$\|e^{ne^{i\theta_0}A}\| \le C_0 e^{(1-\epsilon_0)n} \tag{2.8}$$

where

$$C_0 := \sup_{\lambda\in\gamma_0}\|(\lambda - A)^{-1}\| l(\gamma_0)$$

$$l(\gamma_0) := \frac{1}{2\pi}\int_{\gamma_0}|d\lambda|.$$

We claim that there is a $\delta > 0$ such that for $|\theta - \theta_0| < \delta$

$$\|e^{ne^{i\theta}A}\| \le 2C_0 e^{(1-\epsilon_0)n} \tag{2.9}$$

In fact, we can integrate along γ consisting of points λ such that $e^{i(\theta-\theta_0)}\lambda \in \gamma_0$. For small enough δ γ is still a contour around the spectrum and by construction $\Re\{e^{i\theta}\lambda\} \leq 1-\epsilon_0$. Further, we may also assume that δ is small enough that

$$\sup_{\lambda\in\gamma} \|(\lambda - A)^{-1}\| \leq 2 \sup_{\lambda\in\gamma_0} \|(\lambda - A)^{-1}\|.$$

Now, by (2.9) we have an open cover for K and we can choose from a finite subcover a largest C and smallest ϵ such that

$$\|e^{ne^{i\theta}A}\| \leq 2Ce^{(1-\epsilon)n} \tag{6.7}$$

holds for all $n \geq 0$ and for all $e^{-i\theta} \in K$.

Returning to the integral (2.4) we can estimate the contribution of $z/n \in K$. We obtain exponential decay:

$$\|\frac{n!}{2\pi i}\int_{nK} z^{-1-n}e^{zA}dz\| \leq 2Ce^{(1-\epsilon)n}\frac{n!}{n^n} = \mathcal{O}(\sqrt{n}e^{-\epsilon n}).$$

Proof of Theorem 8.

Assume (1.29) holds. Then from the exponential formula

$$e^{te^{i\varphi}A} = \lim(1 - \frac{te^{i\varphi}}{n}A)^{-n}$$

we obtain (1.30). The reverse direction follows by writing

$$(r - e^{-i\varphi}A)^{-1} = \int_0^\infty e^{-tr}e^{te^{-i\varphi}A}dt$$

Differentiating this $n-1$ times and using (1.30) implies (1.29).

Suppose now that A_ϵ is power bounded with the constant C_1. Then we have by Theorem 7

$$\|e^{zA_\epsilon}\| \leq C_1e^{|z|}$$

which can be written as

$$\|e^{(1+\epsilon)te^{i\varphi}A}e^{-\epsilon te^{i\varphi}}\| \leq C_1e^t.$$

This implies for $|\varphi| \leq \pi$

$$\|e^{te^{i\varphi}A}\| \leq C_1e^{t(1-\frac{\epsilon}{1+\epsilon}\frac{\varphi^2}{\pi^2})}.$$

Next we show that if A satisfies (1.30) then A_ϵ satisfies the same inequality with constants C and $c/2$ if $0 \leq \epsilon \leq \frac{c}{1-2c}$. In fact, we have

$$\|e^{te^{i\varphi}A_\epsilon}\| = \|e^{(1+\epsilon)te^{i\varphi}A}e^{-\epsilon te^{i\varphi}}\| \leq Ce^{(1+\epsilon)t(1-c\varphi^2)-\epsilon t(1-\varphi^2/2)} \leq Ce^{t(1-c\varphi^2/2)}.$$

It suffices to show that if (7.2) holds then A is power bound as then by the previous result the same holds for A_ϵ. We use the formula

$$A^n = \frac{n!}{2\pi i}\int z^{-1-n}e^{zA}dz$$

and take $|z| = n$. Thus

$$\|A^n\| \le Cn!(\frac{e}{n})^n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-cn\varphi^2} d\varphi \le \frac{C}{\sqrt{c}}\sqrt{2\pi(n+1)}E_0/\sqrt{n}$$

where

$$E_k = \frac{1}{\pi}\int_0^\infty t^k e^{-t^2} dt.$$

Finally, the identity

$$A^n(A-1) = \frac{n!}{2\pi i}\int (\frac{n+1}{z} - 1) z^{-1-n} e^{zA} dz$$

gives choosing $|z| = n+1$ the other inequality in the same manner:

$$\|A^n(A-1)\| \le \frac{C}{c} E_1 \frac{\sqrt{2\pi(n+2)}}{n+1}.$$

Proof of Theorem 9.

Let us start with a lemma.

Lemma.
If $\sigma(A) \subset \overline{\mathbb{D}}$ and for $|\lambda| = 1$ we have

$$\|(\lambda - A)^{-1}\| \le \frac{C}{|\lambda - 1|^{1+\alpha}}$$

then there exists a curve $\gamma = \gamma(\varphi)$ such that

$$\gamma(\varphi) = e^{i\varphi - c\varphi^{1+\alpha}}$$

holds with some $c > 0$ and for $|\varphi| \le \varphi_0$,

$$|\gamma(\varphi)| \le 1 - \delta$$

for $\varphi_0 \le |\varphi| \le \pi$, $\gamma(-\pi) = \gamma(\pi)$ and for $\lambda \in Ext(\gamma)$ such that $|\lambda| \le 2$ we have

$$\|(\lambda - A)^{-1}\| \le \frac{C_1}{|\lambda - 1|^{1+\alpha}}$$

with some C_1.

Proof of Lemma. Let $\Omega := \{\lambda | \lambda \in Ext(\gamma) \text{ and } |\lambda| \le 2\}$. Then $u(\lambda) := |\lambda - 1|^{1+\alpha}\|(\lambda - A)^{-1}\|$ is subharmonic in Ω (provided $\sigma(A) \subset Int(\gamma)$). In fact, $f(\lambda) := (\lambda - 1)^{1+\alpha}(\lambda - A)^{-1}$ is analytic outside the unit disc and so u must be subharmonic in Ω as u is just the norm of some analytic continuation of f into Ω.

All we need to do is to conclude that there exists γ of the given form such that u is bounded along it because u is bounded along $|\lambda| = 2$ and therefore the result follows from the maximum principle.

Let $\pi \geq \varphi \geq 0$ and

$$|\mu| \leq \frac{1}{2}\frac{|e^{i\varphi}-1|^{1+\alpha}}{C}$$

then

$$\begin{aligned}||e^{i\varphi}-\mu-A)^{-1}|| &\leq \frac{C}{|e^{i\varphi}-1|^{1+\alpha}}\frac{1}{1-|\mu|\frac{C}{|e^{i\varphi}-1|^{1+\alpha}}} \\ &\leq \frac{2C}{|e^{i\varphi}-1|^{1+\alpha}} \\ &\leq \frac{C_1}{\varphi^{1+\alpha}}\end{aligned}$$

for small $|\varphi|$. The claim follows.

For each $n \geq 1$ we shall define an integration path γ_n to consist of three parts as follows:

$$\gamma_n = \gamma_{n,A} \cup \gamma_{n,B} \cup \gamma_{n,C}.$$

Let $\beta > 0$ be small enough so that

$$(1+\frac{\beta}{n})|\gamma(\varphi)| \leq 1-\delta/2$$

for $\varphi_0 \leq |\varphi| \leq \pi$. Then put

$$\gamma_n(\varphi) := (1+\frac{\beta}{n})\gamma(\varphi).$$

For small enough c, independent of n we call the part of γ_n as $\gamma_{n,A}$ when $|\varphi| \leq \varphi_n := (\frac{c}{n})^{\frac{1}{1+\alpha}}$, while $\gamma_{n,B}$ corresponds to $\varphi_n \leq |\varphi| \leq \varphi_0$ and $\gamma_{n,C}$ to $|\varphi| \geq \varphi_0$.

Then we divide the integration into three parts according to the division of γ_n:

$$||A^n(A-1)^k|| = ||\frac{1}{2\pi}\int_{\gamma_n}\lambda^n(\lambda_1)^k(\lambda-A)^{-1}d\lambda|| =: ||I_{n,A}+I_{n,B}+I_{n,C}||.$$

Here

$$||I_{n,C}|| \leq (1-\delta/2)^n\frac{1}{2\pi}\int_{\gamma_{n,C}}||(\lambda-1)^k(\lambda-A)^{-1}|||d\lambda| = M_k(1-\delta/2)^n.$$

Likewise

$$||I_{n,B}|| \leq Const(1+\beta/n)^n\frac{1}{\pi}\int_{\varphi_n}^{\varphi_0}e^{-n\varphi^{1+\alpha}}\varphi^{k-1-\alpha}d\varphi$$

as $|\lambda-1|$ behaves like φ on that interval. A change of the varible to $\tau = n\varphi^{1+\alpha}$ yields a bound of the form $Const(\frac{1}{n})^{\frac{k-\alpha}{1+\alpha}}$.

Finally, we use the Kreiss condition. Choosing a small enough c we have $|\lambda|-1 \geq (1+\beta/n)e^{-c\varphi_n^{1+\alpha}}-1 \geq const/n$ which gives

$$||(\lambda-A)^{-1}|| \leq \frac{C}{|\lambda|-1|} \leq Const\ n.$$

This yields

$$||I_{n,A}|| \leq Const\ n\int_0^{\varphi_n}\varphi^k d\varphi \leq Const\ n(\frac{d}{n})^{\frac{k-\alpha}{1+\alpha}}.$$

Proof of Theorem 11.

Assuming (1.37) implies that A is power bounded for positive integers. Write $A = 1+L$. Assume then that $\lambda < -1$ is such that (1.40) holds. Put $\alpha := -1/(\lambda - 1)$ so that (1.40) reads

$$||(1+\alpha L)^{-n}|| \leq C_1$$

for $n \geq 0$. Since $0 < \alpha < 1$ we conclude from the power boundedness of $1 + L$ that $1 + \alpha L$ is power bounded, too and thus it is bounded for both positive and negative integers.

From (1.37) we conclude that the sepctrum of $1 + \alpha L$ is in a cone pointing inside into the unit disc at 1. However, as the operator is also bounded on negative integers the spectrum cannot intersect the inside of the disc and we conclude that $\sigma(L) = 0$. However, then an old result of Gelfand [G] implies that $L = 0$.

Proof of Theorem 14.

Since

$$T(1) < \gamma = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \frac{1}{|e^{i\theta}-1|} d\theta$$

we conclude from

$$||(e^{i\theta} - A)^{-1}|| \geq \sup_{\lambda \in \sigma(A)} \frac{1}{|e^{i\theta} - \lambda|}$$

that $\sigma(A)$ is strictly inside the unit disc and in particular $M(1)$ is finite.

Without loss of generality we can assume that $f(\theta) := ||(e^{i\theta} - A)^{-1}||$ assumes its maximum $M(1)$ at $\theta = 0$.

But then we have from $(1 - A)^{-1} = (e^{i\theta} - A)^{-1}(1 - (e^{i\theta} - 1)(1 - A)^{-1})$ that

$$f(\theta) \geq \frac{M(1)}{1 + M(1)|e^{i\theta} - 1|}.$$

Now we estimate as follows (here $M = M(1)$)

$$\begin{aligned} T(1) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ f(\theta) d\theta \\ \geq& \frac{1}{\pi}\int_0^{\pi/3} \log \frac{1}{|e^{i\theta}-1|} d\theta - \frac{1}{\pi}\int_0^{\pi/3} \log(1 + \frac{1}{M|e^{i\theta}-1|}) d\theta \\ \geq& \gamma - \frac{1}{3}\log(1 + 1/M) - \frac{1}{3M}\log(M+1) \end{aligned}$$

which implies the claim.

References

[A] A.Atzmon: Operators which are annihilated by analytic functions and invariant subsapces, Acta Math. 144 (1980), 27-63

[B] N.Borovykh, D.Drissi, M.N.Spijker: A note about Ritt's condition and related resolvent conditions, Report TW-98-10, Leiden University, Department of Mathematics and Computer Science

[G] I.Gelfand: Zur Theorie der Charactere der Abelschen topologischen Gruppen, Mat.Sb. 9 (1941), 49-50

[K] Y.Katznelson, L.Tzafriri: On power bounded operators, J. Funct. Anal.68 (1986), 313-328

[Le] R.J. LeVeque, L.N. Trefethen: On the resolvent condition in the Kreiss matrix theorem, BIT24 (1984), 584-591

[Ly] Yu.Lyubich: Spectral localization, power boundedness and invariant subspaces under Ritt's type condition, Stud.Math. (1999), 134(2), 153-167

[Na] B.Nagy, J.Zemanek: A resolvent condition implying power boundedness, Stud.Math. (1999), 134(2), 143-151

[Ne1] O. Nevanlinna: Convergence of Iterations for Linear Equations, Birkhäuser, 1993.

[Ne2] O. Nevanlinna: Meromorphic resolvents and power bounded operators, BIT 36: 3 (1996), pp. 531-541.

[Ne3] O. Nevanlinna: Convergence of Krylov methods for sums of two operators, BIT 36: 4 (1996), pp. 775-785.

[Ne4] O. Nevanlinna: Growth of operator valued meromorphic functions, to appear in Ann.Acad. Sci. Fenn. Ser A. Math.

[Ne5] O. Nevanlinna: On the growth of the resolvent operators for power bounded operators, Banach Center Publications, Vol. 38 (1997), pp. 247-264.

[R] R.K.Ritt, A condition that $\lim_{n\to\infty} n^{-1}T^n = 0$, Proc.Amer.Math.Soc. 4 (1953),898-899

[Sh] A.Shields, On Möbius bounded operators, Acta Sci.Math. 40,(1978), 371-374

[Sp] M.N.Spijker, S.Tracogna, B.D. Welfert: About the sharpness of the stability estimates in the Kreiss matrix theorem, Report TW-98-02, Leiden University, Department of Mathematics and Computer Science

[SW] J.Strikwerda, B.Wade: A survey of the Kreiss matrix theorem for power bounded families of matrices and its extensions, Banach Center Publications, Vol. 38 (1997), pp. 339-360.

[T] E.Tadmor: The resolvent condition and uniform power boundedness, Lin.Alg.Appl. 80 (1986), 250-252