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IMPLICIT FUNCTIONS FROM LOCALLY CONVEX SPACES TO BANACH SPACES

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Abstract: *We generalize the classical implicit function theorem of Hildebrandt and Graves to the case where we have a k times continously Keller Π-differentiable map, defined on an open subset of the product of locally convex space and a Banach space, and with values in the same Banach space. As an application, we prove that under certain transversality condition the preimage of a submanifold is a submanifold for a map from a Fréchet manifold to a Banach manifold.*

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Implicit functions from locally convex spaces to Banach spaces

by

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Abstract. We generalize the classical implicit function theorem of Hildebrandt and Graves to the case where we have a Keller C^k_Π–map f, defined on an open subset of the space $E \times F$ and with values in F, for E an arbitrary Hausdorff *locally convex space* and F a *Banach space.* As an application, we prove that under a certain transversality condition the preimage of a submanifold is a submanifold for a map from a Fréchet manifold to a Banach manifold.

0. Introduction and preliminaries.

Our main objective is the following

IMPLICIT FUNCTION THEOREM. *Let E and F be locally convex spaces with F Banach. Assuming $k \in \mathbb{N} \cup \{\infty\}$, let $f : E \times F \supseteq \operatorname{dom} f \to F$ be a C^k–map with $f(x,y) = z$. If $\partial_2 f(x,y) : F \to F$ is bijective, there exist open sets U and V in the spaces E and F, respectively, such that $(x,y) \in U \times V \subseteq \operatorname{dom} f$ and the set $f^{-1}(z) \cap (U \times V)$ is a C^k–map $E \supseteq U \to F$.*

Here $\mathbb{N}$ is the set of positive integers and $\operatorname{dom} f$ is the domain set of the function f. By definition 0.8 below, a C^k–map always has open domain. Based on the above implicit function theorem, we then prove as a corollary the following

THEOREM. *Assuming $k \in \mathbb{N} \cup \{\infty\}$, let M and N be C^k–manifolds with M modelled on Fréchet and N on Banach spaces. Let $f : M \supseteq \operatorname{dom} f \to N$ be a C^k–map. If S is a C^k–submanifold of N and for all $(y,x) \in f^{-1}|S$ conditions* (1) *and* (2) *below are satisfied, then $f^{-1}(S)$ is a C^k–submanifold of M.*

(1) $\pi \circ Tf_x : T_xM \to T_yN \to T_yN/T_yS$ *is surjective*

(2) $\operatorname{Ker}(\pi \circ Tf_x)$ *is complemented in* T_xM

Here $f^{-1}|S$ is the relation f^{-1} restricted to the set S. The π above is the quotient map $T_yN \to T_yN/T_yS$. If T_yN/T_yS is finite dimensional, then the requirement (2) is superfluous. We treat real and complex scalars simultaneously. Consequently, the holomorphic case is also included; cf. Remarks 0.12 below and [6; Lemma 2.6].

Now we explain our notion C^k of continuous differentiability, which for *real scalars* is a weakened adaptation of the concept C_k of [1], and which for maps between locally convex spaces coincides with C^k_Π of [4].

Our fundamental category of maps is that formed by all separated limit (or convergence) spaces as objects and all continuous functions from *some open subset* of the domain space to the range space. A limit space X is a set S endowed with a convergence (structure; in [1] a pseudo-topology) Λ, $X = (S, \Lambda)$. It is separated iff for any filter Φ on S converging to both x and y, we have $x = y$. A set $V \subseteq S$ is open iff $V \in \Phi$ for every filter Φ converging to any $x \in V$. A function is continuous iff it maps convergent filter bases in its domain set to convergent filter bases in its range space. A filter base in a subset of a limit space is convergent iff the filter generated by it on the whole space is convergent. We really obtain a category by

0.1 LEMMA. *Let X, Y be convergence spaces and $f : X \supseteq \operatorname{dom} f \to Y$ a continuous function, with* $\operatorname{dom} f$ *open in X. If B is open in Y, then $A = f^{-1}(B)$ is open in X.*

Proof. If a filter Φ in X converges to $x \in A$, then $\{\, U \cap \operatorname{dom} f : U \in \Phi \,\}$ is a filter base in $\operatorname{dom} f$, converging to x. Thus the filter $\Psi = \{V : \exists\, U \in \Phi; f(U \cap \operatorname{dom} f) \subseteq V \subseteq Y\}$ converges to $f(x)$ in Y. Hence $B \in \Psi$. So, for some $U \in \Phi$, we have $f(U \cap \operatorname{dom} f) \subseteq B$. Thus $U \cap \operatorname{dom} f \subseteq f^{-1}(B) = A$. Hence $A \in \Phi$, due to $\operatorname{dom} f \in \Phi$. We are ready, because Φ and x are arbitrary. □

As an embedded full subcategory, we consider continuous maps between separated real ($\mathbb{K} = \mathbb{R}$) or complex ($\mathbb{K} = \mathbb{C}$) convergence vector spaces. In [1], these are called pseudo-topological vector spaces. To every convergence space we have adjoined a topology in the above way. Conversely, a topology defines a convergence. A topological vector space can thus be interpreted as a convergence vector space. For the basic facts about convergence vector spaces, see, e.g., [1] or [2].

For a convergence vector space E, denote by $\mathcal{N}_o E$ the neighbourhood filter of zero in the topology of E. Put $\mathcal{V}_o = \mathcal{N}_o \mathbb{K}$. A filter Φ on a vector space X is called equable iff $\Phi = [\mathcal{V}_o\Phi]$. Here $[\mathcal{V}_o\Phi]$ is the filter on X generated by the filter base $\mathcal{V}_o\Phi$ formed by the sets $VB = \{\lambda x : \lambda \in V$ and $x \in B\}$, where $V \in \mathcal{V}_o$ and $B \in \Phi$. A convergence vector space E is called *equable* iff for every zero filter (i.e., converging to zero) Φ there exists a smaller (i.e., $\Psi \subseteq \Phi$) equable zero filter Ψ. With every convergence vector space E, we associate the equable space E^{eq} with the same underlying vector space and as zero filters those filters on E for which there is a smaller equable zero filter. Every topological vector space E is equable, because $\mathcal{N}_o E$ is equable and contained in every zero filter. For more about equable spaces, see [1]. There E^{eq} is denoted by $E^{\#}$.

To a pair E, F of convergence vector spaces, we adjoin the equable space $\mathcal{L}_{eq}(E, F)$ of continuous linear mappings as follows. Call a filter ζ on E *quasi bounded* iff $[\mathcal{V}_o\zeta]$ is a zero filter in E. Let $M = \mathcal{L}(E, F)$ be the vector space of continuous linear maps $E \to F$ (defined on all of E). Adjoin to M the unique vector convergence, the zero filters of which are those filters Φ on M satisfying the condition, that for all quasi bounded ζ in E, the filter $[\Phi\zeta]$ on F is a zero filter in F. In this way, we obtain the convergence vector space $\mathcal{L}_{qb}(E,F)$. Now put $\mathcal{L}_{eq}(E,F) = (\mathcal{L}_{qb}(E,F))^{eq}$. In [1] the space $\mathcal{L}_{eq}(E,F)$ is denoted by $L^{\#}(E;F)$. For locally convex spaces E, F, we have $\mathcal{L}_{eq}(E,F) = \mathcal{L}_{\Pi}(E,F)$, where the latter space is that considered in [4]. By [4; p. 56, Corollary 0.7.4], we have

0.2 Lemma. *Let G, F be locally convex spaces, with F normable. Then a filter Φ on $\mathcal{L}(G,F)$ converges to zero in $\mathcal{L}_{eq}(G,F)$ iff*

$$(\Theta) \qquad \exists\, U \in \mathcal{N}_o G\,;\ \forall\, W \in \mathcal{N}_o F\,;\ \exists\, L \in \Phi\,;\ LU \subseteq W.$$

Moreover, the convergence of $\mathcal{L}_{eq}(F,F)$ is given by the norm topology of the normed space $\mathcal{L}(F,F)$ of continuous linear maps $F \to F$.

For a proof, note that condition (Θ) above is equivalent to the requirement that the filter Φ is a zero filter in the space $\mathcal{L}_{\Theta}(G,F)$ of [4].

Let E be a convergence vector space. By a *differentiable curve* in E, we mean a function $c : [0,1] \to E$, which is continuous at the points 0 and 1 and, moreover, is such that for $0 < t < 1$ there exists $c'(t) \in E$ such that the filter base

$$\{\{(s-t)^{-1}(c(s) - c(t)) : 0 \le s \le 1 \text{ and } 0 < |s-t| < \delta\} : \delta > 0\}$$

converges to $c'(t)$ in E. By separatedness, $c'(t)$ is unique. The function $c' :]0,1[\, \ni t \mapsto c'(t) \in E$ is the *derivative* of c.

0.3 Lemma. *Let E be a locally convex space and U a closed convex set in E. If c is a differentiable curve in E with $c(0) = 0_E \in U$ and* $\operatorname{rng} c' \subseteq U$, *then* $\operatorname{rng} c \subseteq U$, *in particular*, $c(1) \in U$.

Proof. Without restriction, assume real scalars. If the claim is false, then $c(s) \notin U$ for some $s \in\,]0,1]$. By Hahn–Banach, for some $\ell \in \mathcal{L}(E, \mathbb{R})$ we have $\ell(U) \subseteq\,]-\infty, 1[$ and $\ell(c(s)) = 1$. The function $\ell \circ c$ satisfies the requirements of the classical intermediate value theorem. Consequently, for some $t \in\,]0,s[$ we get $1 \le s^{-1} = s^{-1}\ell(c(s)) = (\ell \circ c)'(t) = \ell(c'(t)) \in \ell(U) \subseteq\,]-\infty, 1[$, a contradiction. □

Our principal concept of differentiability for maps $f : E \supseteq \operatorname{dom} f \to F$ between locally convex, or, more generally, between (equable) convergence vector spaces, is FB–differentiability, originally introduced in [1]. As auxiliary concepts we need G– and MK–differentiabilities. Here the letters refer to Gâteaux, Michal and Keller. First we define the corresponding "smallness" or remainder concepts.

0.4 Definitions. Let E, F be convergence vector spaces and $r : E \to F$ a function (defined on all of E). Let X denote any of the symbols G, MK or FB. We say that $r : E \to F$, or the triple $\tilde{r} = (E, F, r)$, is X–*small* iff the corresponding condition below is satisfied. For MK we make the restriction that E, F have to be topological vector spaces.

(G) For all $h \in E$, the filter base $\{\{t^{-1}r(th) : 0 < |t| < \delta\} : \delta > 0\}$ converges to 0_F in F, i.e., $\lim_{t\to 0} t^{-1}r(th) = 0_F$.

(MK) $\exists\, U \in \mathcal{N}_o E \;;\; \forall\, W \in \mathcal{N}_o F \;;\; \exists\, V \in \mathcal{N}_o E \;;\; \forall\, t \in I\!K \setminus \{0\}, h \in U \;;\; th \in V \;\Rightarrow\; t^{-1}r(th) \in W$.

(FB) For all quasi bounded ζ in E, the filter base $\{\{t^{-1}r(th) : 0 < |t| < \delta \text{ and } h \in B\} : \delta > 0 \text{ and } B \in \zeta\}$ converges to 0_F in F.

For a convergence vector space G, if $r : E \to F$ is X–small and $b \in \mathcal{L}(F, G)$, then also $b \circ r : E \to G$ is X–small (for MK the above restriction). Trivially, FB–smallness implies G–smallness. Using 0.5 below, we get MK $\Rightarrow$ FB $\Rightarrow$ G. So G–smallness is the weakest.

0.5 Proposition. *If $\tilde{r}$ is* MK – *small, then $\tilde{r}$ is* FB – *small.*

Proof. Let $\tilde{r} = (E, F, r)$ be MK–small. Thus E, F are topological vector spaces, and 0.4(MK) holds. Given quasi bounded ζ in E, we have to prove that for all $W \in \mathcal{N}_o F$ there exist $\delta > 0$ and $B \in \zeta$ such that the implication $0 < |t| < \delta \;\Rightarrow\; t^{-1}r(th) \in W$ holds for $t \in I\!K, h \in B$.

By $\mathcal{N}_o E \subseteq [\,\mathcal{V}_o \zeta\,]$, for the U in 0.4(MK), we have $\varepsilon B_1 \subseteq U$ for some $\varepsilon > 0$ and $B_1 \in \zeta$. Given $W \in \mathcal{N}_o F$, take εW as the W in 0.4(MK). Again using $\mathcal{N}_o E \subseteq [\,\mathcal{V}_o \zeta\,]$, for the V, now given by 0.4(MK), we find $\delta > 0$ and $B_2 \in \zeta$ satisfying $tB_2 \subseteq V$ for $|t| < \delta$. Writing $B = B_1 \cap B_2$, we obtain $B \in \zeta$. Let now $t \in I\!K, h \in B$ be arbitrary with $0 < |t| < \delta$. For $s = \varepsilon^{-1}t$ and $k = \varepsilon h$, we have $k \in U$ and $sk = th \in V$. Consequently, $t^{-1}r(th) = \varepsilon^{-1}(s^{-1}r(sk)) \in \varepsilon^{-1}(\varepsilon W) = W$. □

From now on, by a *vector map* we mean a triple $\tilde{f} = (E, F, f)$ such that E, F are convergence vector spaces and f is a function defined on some subset of E and with range included in F, i.e., $f \subseteq E \times F$. Instead, we may use the phrase "map $f : E \supseteq \operatorname{dom} f \to F$".

0.6 Definitions. Let $\tilde{f} = (E, F, f)$ be a vector map and let X denote any of the symbols G, MK, FB. Then a pair (ℓ, r) is called an X–*expansion* of $\tilde{f}$ at x iff x is an interior point of $\operatorname{dom} f$ in the topology of E, $\ell \in \mathcal{L}(E, F)$, $r : E \to F$ is X–small, and $f(x + h) = f(x) + \ell(h) + r(h)$ holds for $x + h \in \operatorname{dom} f$.

If X = MK, we require E, F to be topological vector spaces. Using a standard argument, the linear mapping ℓ is seen to be unique. The map $\tilde{f}$ is said to be X–*differentiable at* x iff there exists an X–expansion at x. A map is X–differentiable iff it is X–differentiable at every point in its domain set. Then the domain is necessarily open. Mere differentiability will from now on refer to FB–differentiability.

The (Gâteaux) *derivative function* f' of the map $\tilde{f}$ is defined to be the set of all pairs (x, ℓ), such that for some r the pair (ℓ, r) is a G–expansion of $\tilde{f}$ at x. Then the function $f' : \operatorname{dom} f' \to \mathcal{L}(E, F)$ is defined at every point at which $\tilde{f}$ is G–, FB– or MK–differentiable.

The (Frölicher–Bucher) *derivative* (map) of a vector map $\tilde{f}$ is the vector map $D\tilde{f} = (E, \mathcal{L}_{eq}(E,F), f'|A)$ having as domain set the subset A of $\operatorname{dom} f'$ formed by the points at which $\tilde{f}$ is FB–differentiable. If $\tilde{f}$ is differentiable, then $A = \operatorname{dom} f' = \operatorname{dom} f$.

Following the conventional customs and not too severe logical pedantry, we may write Df instead of $D\tilde{f}$, and also f instead of $\tilde{f} = (E, F, f)$.

0.7 Proposition. *Let E, F be locally convex spaces, with F normable, and let the map $\tilde{f} = (E, F, f)$ be* G–*differentiable. If f' is continuous $E \supseteq \operatorname{dom} f \to \mathcal{L}_{eq}(E,F)$, then $\tilde{f}$ is* MK–*differentiable.*

Proof. In view of Lemma 0.2, the claim follows from [4; p. 76, Th. 1.2.11]. □

Let now C_o be the (proper) class (not a set) consisting of all *continuous* vector maps (E, F, f), where E, F are *equable* and $\operatorname{dom} f$ is *open* in E. Putting $I\!N_o = \{0\} \cup I\!N$ and $\infty + 1 = \infty$, we then construct our *differentiability classes* C^k for $k \in I\!N_o \cup \{\infty\}$, as follows.

0.8 Definition. A vector map f belongs to C^k iff there are maps $f^{(l)}$ in C_o for $l \in I\!N_o, l < k+1$, such that $f^{(0)} = f$ and such that $f^{(i)}$ is differentiable with $f^{(i+1)} = Df^{(i)}$ for $i \in I\!N_o, i < k$.

The usual recursivity of continuous differentiability holds. That is, for any $k \in I\!N_o \cup \{\infty\}$ we have the equivalences: $f \in C^{k+1} \Leftrightarrow [\, f \in C^1$ and $Df \in C^k\,] \Leftrightarrow [\, f$ differentiable and $f, Df \in C^k\,]$.

Consider maps between equable spaces. By a trivial induction, one sees that constants, defined on an open subset of the domain space, are in C^∞. This implies, by the above recursivity, that continuous linear maps are in C^∞. To prove that continuous bilinear maps are in C^∞, using [1; p. 44, 4.2.3], one only has to show the continuity of the derivative. By [1; Propositions 6.3.3, p. 72 and 2.8.3, p. 24], the bilinear composition map $\operatorname{comp} : \mathcal{L}_{eq}(E,F) \times \mathcal{L}_{eq}(F,G) \ni (k,\ell) \mapsto \ell \circ k \in \mathcal{L}_{eq}(E,G)$ is continuous, hence in C^∞.

For maps $\tilde{f} = (E, F, f)$ and $\tilde{g} = (F, G, g)$ in $C^0 = C_o$, we define the composition $\tilde{f}\tilde{g}$ as $(E, G, g \circ f)$. Here $\operatorname{dom}(g \circ f) = f^{-1}(\operatorname{dom} g)$ is open in E by Lemma 0.1. If $\tilde{f}, \tilde{g} \in C^1$, then $\tilde{f}\tilde{g}$ is differentiable and the first order *chain rule formula* $(g \circ f)' = \operatorname{comp} \circ [f', g' \circ f]$ holds, see [1; pp. 38–41]. In general, for maps $\tilde{f}_i = (E, F_i, f_i)$ where $i = 1, 2$, we define the map $[\tilde{f}_1, \tilde{f}_2]$ as $[f_1, f_2] : E \supseteq (\operatorname{dom} f_1) \cap (\operatorname{dom} f_2) \ni x \mapsto (f_1(x), f_2(x)) \in F_1 \times F_2$, cf. [1; p. 3, 1.3.2].

Each class C^k forms a *category* under the composition $(\tilde{f}, \tilde{g}) \mapsto \tilde{f}\tilde{g}$ defined above. This follows from a general order k *chain rule*, Proposition 0.11 below. We also get *functors* $T : C^{k+1} \to C^k$ by forming *tangent maps*; for $\tilde{f} = (E, F, f)$ put $T : \tilde{f} \mapsto T\tilde{f} = (E \times E, F \times F, Tf)$, where in the sense of [4]. Our concept C^k is, in general, weaker than the concept C_k, considered in [1]. In this respect, there is some confusion in [4], because Keller seems to think that they are equivalent for maps between locally convex spaces. However, this is not the case even for scalar valued maps defined on the whole of an infinite dimensional Hilbert space. Cf. [4; p. 11, (2), p. 74, 1.2.7. Remark, p. 97, 2.6.3. Remarks (2)].

Proceeding by induction, using Lemma 0.2, [1; p. 42, Prop. 4.1.1], and the recursive property stated after 0.8, one can directly prove that for normable spaces E, F, we have $(E, F, f) \in C^k$ iff f is a C^k–function $E \supseteq \operatorname{dom} f \to F$ in the classical sense.

0.13 REMARKS. Denote by $\mathcal{L}is\,(E, F)$ the set of linear homeomorphisms $E \to F$. By the open mapping theorem, for *Fréchet* spaces E, F, we have $\ell \in \mathcal{L}is\,(E, F)$ iff ℓ is a continuous linear isomorphism $E \to F$. Later we need to know that for a Banach space F, the map inv : $\mathcal{L}_{eq}(F, F) \supseteq \mathcal{L}is\,(F, F) \ni \ell \mapsto \ell^{-1} \in \mathcal{L}_{eq}(F, F)$ is in C^∞. This follows from the last part of 0.12 by recalling that in Banach space calculus inv is a C^∞–map. In particular, note that the set $\mathcal{L}is\,(F, F)$ is open in $\mathcal{L}_{eq}(F, F)$.

We also need to know that for maps $(E_1, F, f), (E_2, G, g) \in C^k$, the map $f \times g : E = E_1 \times E_2 \supseteq (\operatorname{dom} f) \times (\operatorname{dom} g) \ni (x, y) \mapsto (f(x), g(y)) \in F \times G$ is in C^k. This follows from 0.10 and 0.11, because we can write $f \times g = [f \circ p, g \circ q]$, where $p : E \to E_1$ and $q : E \to E_2$ are natural continuous linear projections.

1. Implicit Function Theorem. Let E, F be locally convex spaces. Let the topology of F be given by a Banach norm $x \mapsto \|x\|$. Put $G = E \times F$. Let $\mathrm{B}(x, \delta)$, $\mathrm{B}(\delta)$ and $\bar{\mathrm{B}}(\delta)$ be the open and closed balls in F with radius δ and centered at x and zero, respectively. The partial derivatives below are derivatives of the corresponding partial maps.

1.1 LEMMA. *Let the map $\varphi : G \supseteq \operatorname{dom} \varphi \to F$ be in C^1, with φ and φ' mapping zero to zero. Then for every $\ell \in \mathcal{L}(E, F)$ there exists a continuous seminorm p in E, an open neighbourhood U of zero in E, and $\delta > 0$, such that for $V = \mathrm{B}(\delta)$ we have $U \times V \subseteq \operatorname{dom} \varphi$ and the set $g = \{ (x, y) \in U \times V : \varphi(x, y) + \ell(x) = y \}$ is a function with* $\operatorname{dom} g = U$, $\operatorname{rng} g \subseteq V$, *and satisfying $\|g(u) - g(v)\| \leq p(u - v)$ for all $u, v \in U$.*

Proof. Put $\Phi = \{ B : \exists\, A \in \mathcal{N}_o G;\ \varphi'[A] \subseteq B \subseteq \mathcal{L}(G, F) \}$. Then, by the assumptions, Φ is a zero filter in $\mathcal{L}_{eq}(G, F)$. Applying Lemma 0.2, let $W_u \in \mathcal{N}_o G$ be the U given by condition (Θ).

The linear mapping ℓ being continuous, we have $\ell(U_1) \subseteq \mathrm{B}(1)$ for some absolutely convex $U_1 \in \mathcal{N}_o E$. We can assume $U_1 \times \mathrm{B}(\delta_1) \subseteq W_u$ for some $\delta_1 > 0$. Let p_1 be the Minkowski functional of U_1. Put $p = 6p_1$ to obtain the continuous seminorm searched for. Let $q : G \ni (x, y) \mapsto p_1(x) + \frac{1}{2}\|y\|$.

For $\alpha = \inf\{\frac{1}{2}, \frac{1}{3}\delta_1\}$, let $\mathrm{B}(\alpha)$ be the W in condition $0.2(\Theta)$. Then, for some $U_0 \in \mathcal{N}_o E$ and some $\delta > 0$ we have $\|\varphi'(z)w\| < \alpha$ for $z \in U_0 \times \mathrm{B}(\delta)$ and $w \in W_u$. By the openness of $\operatorname{dom}\varphi$, we can assume $U_0 \subseteq U_1$ and $U_0 \times \mathrm{B}(\delta) \subseteq \operatorname{dom}\varphi$.

By the continuity of $\psi : \operatorname{dom}\varphi \ni (x,y) \mapsto \varphi(x,y) + \ell(x)$ we find an absolutely convex open neighbourhood U of zero in E with $U \subseteq U_0$, such that $\|\psi(x, 0_F)\| < \frac{1}{3}\delta$ for $x \in U$. Write $V = \mathrm{B}(\delta)$ and $W = U \times V$. Then $W \subseteq U_0 \times \mathrm{B}(\delta) \subseteq \operatorname{dom}\varphi$.

Next, we consecutively state and prove claims (1) through (6) below. From these claims we get our Lemma.

(1) $\forall\ z \in W, w \in G\,;\ \|\varphi'(z)w\| \leq q(w)$. To prove this, let $z \in W$, $w = (u,v) \in G$, and assume $s = q(w) \neq 0$ for the beginning. We have $p_1(\frac{\alpha}{s}\,u) = \frac{\alpha}{s}\,p_1(u) \leq \frac{\alpha}{s}\,(q(w)) = \alpha \leq \frac{1}{2}$ and $\|\frac{\alpha}{s}\,v\| = \frac{\alpha}{s}\|v\| = \frac{2\alpha}{s}\,(\frac{1}{2}\|v\|) \leq \frac{2\alpha}{s}\,(q(w)) = 2\,\alpha \leq \frac{2}{3}\,\delta_1$. Hence $\frac{\alpha}{s}\,u \in U_1$ and $\frac{\alpha}{s}\,v \in \mathrm{B}(\delta_1)$. So $\frac{\alpha}{s}\,w = (\frac{\alpha}{s}\,u, \frac{\alpha}{s}\,v) \in U_1 \times \mathrm{B}(\delta_1) \subseteq W_u$. Then $\|\varphi'(z)w\| = \frac{s}{\alpha}\|\varphi'(z)(\frac{\alpha}{s}\,w)\| < \frac{s}{\alpha}\,\alpha = s = q(w)$. Assuming next $q(w) = 0$, for all $s > 0$, we have $p_1(s\,u) = \|s\,v\| = 0$, hence $s\,w \in U_1 \times \{0_F\} \subseteq W_u$. So $s\|\varphi'(z)w\| = \|\varphi'(z)(s\,w)\| < \alpha$. Thus $\|\varphi'(z)w\| = 0 \leq q(w)$.

(2) $\forall\ w, z \in W\,;\ \|\varphi(w) - \varphi(z)\| \leq q(w-z)$. To prove this, fix w, z and let $c : [0,1] \ni t \mapsto \varphi(z + t\,(w-z)) - \varphi(z) \in F$. Then c is a differentiable curve in F with $c(0) = 0_F$ and $c'(t) = \varphi'(z + t\,(w-z))(w-z)$ for $0 < t < 1$. So, by (1), we have $\|c'(t)\| \leq q(w-z)$. Taking then $\bar{\mathrm{B}}(q(w-z))$ as the U in Lemma 0.3, we get $\|\varphi(w) - \varphi(z)\| = \|c(1)\| \leq q(w-z)$.

(3) $\forall\ x \in U, y \in \bar{\mathrm{B}}(\frac{2}{3}\,\delta)\,;\ \psi(x,y) \in \bar{\mathrm{B}}(\frac{2}{3}\,\delta)$. For a proof, by (2) we first obtain $\|\varphi(x,y) - \varphi(x,0_F)\| \leq q(0_E, y) = \frac{1}{2}\|y\|$. Now recall the relation $\|\psi(x,0_F)\| < \frac{1}{3}\,\delta$, to get $\|\psi(x,y)\| \leq \|\psi(x,y) - \psi(x,0_F)\| + \|\psi(x,0_F)\| = \|\varphi(x,y) + \ell(x) - (\varphi(x,0_F) + \ell(x))\| + \|\psi(x,0_F)\| = \|\varphi(x,y) - \varphi(x,0_F)\| + \|\psi(x,0_F)\| < \frac{1}{2}\|y\| + \frac{1}{3}\,\delta \leq \frac{1}{2}\cdot\frac{2}{3}\,\delta + \frac{1}{3}\,\delta = \frac{2}{3}\,\delta$.

(4) $\forall\ x \in U, y, z \in V\,;\ \|\psi(x,y) - \psi(x,z)\| \leq \frac{1}{2}\|y - z\|$. As in (3), we calculate: $\|\psi(x,y) - \psi(x,z)\| = \|\varphi(x,y) + \ell(x) - (\varphi(x,z) + \ell(x))\| = \|\varphi(x,y) - \varphi(x,z)\| \leq q(0_E, y-z) = \frac{1}{2}\,\|y - z\|$.

By (3) and (4), for a fixed $x \in U$, the function $\{\,(v,z) : (x,v,z) \in \psi\,\}$, i.e., $y \mapsto \psi(x,y)$, is contractive $\bar{\mathrm{B}}(\frac{2}{3}\,\delta) \to \bar{\mathrm{B}}(\frac{2}{3}\,\delta)$. Thus $(x,y,y) \in \psi$ holds for some unique $y \in \bar{\mathrm{B}}(\frac{2}{3}\,\delta) \subseteq \mathrm{B}(\delta) = V$. Then $g = \{\,(x,y) : (x,y,y) \in \psi\,\} \cap (U \times V)$ is a function $U \to V$.

(5) $\forall\ u \in E\,;\ \|\ell(u)\| \leq 2p_1(u)$. For a proof, assuming first $s = p_1(u) \neq 0$, we have $p_1(\frac{1}{2s}\,u) = \frac{1}{2s}\,p_1(u) = \frac{1}{2}$. Hence $\frac{1}{2s}\,u \in U_1$. Thus $\ell(\frac{1}{2s}\,u) \in \mathrm{B}(1)$, and we get $\|\ell(u)\| = 2s\|\ell(\frac{1}{2s}\,u)\| \leq 2\,s = 2\,p_1(u)$. Assuming now $p_1(u) = 0$, for all $s > 0$, we have $s\,u \in U_1$, hence $s\|\ell(u)\| = \|\ell(s\,u)\| < 1$. Thus $\|\ell(u)\| = 0 \leq 2\,p_1(u)$.

(6) $\forall\ u, v \in U\,;\ \|g(u) - g(v)\| \leq p(u-v)$. To prove this, write $x = g(u)$ and $y = g(v)$, so that $x = \psi(u,x) = \varphi(u,x) + \ell(u)$ and $y = \psi(v,y) = \varphi(v,y) + \ell(v)$. By (2) and (5), we obtain $\|x - y\| = \|\varphi(u,x) - \varphi(v,y) + \ell(u-v)\| \leq \|\varphi(u,x) - \varphi(v,y)\| + \|\ell(u-v)\| \leq q(u-v, x-y) + 2\,p_1(u-v) =$

$3\,p_1(u-v)+\frac{1}{2}\|x-y\|$, from which we get $\|g(u)-g(v)\|=\|x-y\|\leq 6\,p_1(u-v)=p(u-v)$. □

1.2 LEMMA. *Let $f : G \supseteq \operatorname{dom} f \to F$ be in C^1, with $f(x_0,y_0)=0_F$ and $\partial_2 f(x_0,y_0)$ bijective $F\to F$. Then there exists a continuous seminorm p in E, an open neighbourhood U of x_0 in E, and $\delta>0$, such that for $V=\mathrm{B}(y_0,\delta)$ and $g=f^{-1}(0_F)\cap(U\times V)$, we have $U\times V\subseteq \operatorname{dom} f$ and g is a function $U\to V$ satisfying $\|g(u)-g(v)\|\leq p(u-v)$ for all $u,v\in U$.*

Proof. Put $\ell_1=\partial_1 f(x_0,y_0)$ and $\ell_2=\partial_2 f(x_0,y_0)$. Then $\ell_1\in\mathcal{L}(E,F)$ and $\ell_2\in\mathcal{L}is\,(F,F)$, by 0.13. Hence $\ell=-\ell_2^{-1}\circ\ell_1\in\mathcal{L}\,(E,F)$. Let the maps $\psi,\varphi : G\supseteq -(x_0,y_0)+\operatorname{dom} f\to F$ be defined by the relations $\psi\,(x,y)=y-(\ell_2^{-1}\circ f)(x_0+x,y_0+y)$ and $\varphi\,(x,y)=\psi(x,y)-\ell(x)$. Then we have $\psi,\varphi\in C^1$ and $\varphi(0_E,0_F)=0_F$.

Trivially, $f^{-1}(0_F)=\{(x_0+x,y_0+y) : (x,y,y)\in\psi\}$. Moreover, for $(u,v)\in G$ we have $\varphi'(0_E,0_F)(u,v)=\psi'(0_E,0_F)(u,v)-\ell(u)$
$=v-\ell_2^{-1}(f'(x_0,y_0)(u,v))+\ell_2^{-1}(\ell_1 u)=v-\ell_2^{-1}(\ell_1 u+\ell_2 v)+\ell_2^{-1}(\ell_1 u)$
$=0_F$. So also φ' maps zero to zero.

By Lemma 1.1, there exists a continuous seminorm p in E, an open neighbourhood U_0 of zero in E, and $\delta>0$, such that the conclusion of 1.1 holds with U replaced by U_0, and g replaced by $g_0=\{\,(x,y)\in U_0\times\mathrm{B}(\delta) : \varphi(x,y)+\ell(x)=y\,\}$. Then for $U=x_0+U_0$, we get our claim, since $f^{-1}(0_F)\cap(U\times V)=\{\,(x_0+x,y_0+y) : (x,y)\in g_0\,\}$. □

1.3 LEMMA. *Let $f : G\supseteq\operatorname{dom} f\to F$ be in C^1, with $f(x_0,y_0)=0_F$ and $\partial_2 f(x_0,y_0)$ bijective $F\to F$. Then there exists a continuous seminorm p in E, an open neighbourhood U of x_0 in E, and $\delta>0$, such that for $V=\mathrm{B}(y_0,\delta)$ we have $U\times V\subseteq\operatorname{dom} f$ and the set $g=f^{-1}(0_F)\cap(U\times V)$ is a continuous function $U\to V$ satisfying*

(1) $\partial_2 f(U\times V)\subseteq\mathcal{L}is\,(F,F)$,

(2) (E,F,g) *is* MK–*differentiable*,

(3) $g'(x)=-\,(\partial_2 f(x,g(x)))^{-1}\circ(\partial_1 f(x,g(x)))$ *for all* $x\in U$.

Proof. For $j_2 : F\ni v\mapsto(0_E,v)\in G$, the partial composition map $\Gamma : \mathcal{L}_{eq}(G,F)\ni\ell\mapsto\ell\circ j_2\in\mathcal{L}_{eq}(F,F)$ is continuous. By $\partial_2 f=\Gamma\circ f'$ and continuity of f', we get continuity of $\partial_2 f : G\supseteq\operatorname{dom} f\to\mathcal{L}_{eq}(F,F)$. Applying Lemma 0.1 and recalling 0.13, we see that $A=(\partial_2 f)^{-1}(\mathcal{L}is\,(F,F))$ is open in G and that $(x_0,y_0)\in A$.

Let p,U,δ,V be those given by Lemma 1.2, when applied to the map $f|A : G\supseteq A\cap(\operatorname{dom} f)\to F$. Then the only non-trivial claims are (2) and (3), which we now prove. For this, let $z_1=(x_1,y_1)\in g$ be arbitrary. The function f' is continuous $G\supseteq\operatorname{dom} f\to\mathcal{L}_{eq}(G,F)$. Thus, by Proposition 0.7, (G,F,f) is MK–differentiable. So, for some MK–small $s : G\to F$, we have $f(z_1+w)=f(z_1)+f'(z_1)w+s(w)$ for $z_1+w\in\operatorname{dom} f$. Write $\ell_1=\partial_1 f(z_1)$ and $\ell_2=\partial_2 f(z_1)$. Then $\ell_2\in\mathcal{L}is\,(F,F)$. Consequently,

$s_1 = -\ell_2^{-1} \circ s$ is MK–small $G \to F$.

Define $r : E \to F$ by $h \mapsto s_1(h, g(x_1 + h) - g(x_1))$ for $x_1 + h \in U$, and by $h \mapsto 0_F$ otherwise. To show $\tilde{r} = (E, F, r)$ to be MK–small, let $U_0 \in \mathcal{N}_o G$ be the U of 0.4(MK) for the map $\tilde{s}_1 = (G, F, s_1)$. Then, for some $U_1 \in \mathcal{N}_o E$ and $\delta_1 > 0$, we have $U_1 \times \mathrm{B}(\delta_1) \subseteq U_0$. Putting now $U_1' = U_1 \cap p^{-1}[0, \delta_1[$, we have $U_1' \in \mathcal{N}_o E$.

To show that U_1' can be taken as the U in 0.4(MK), let $W \in \mathcal{N}_o F$ be arbitrary, and let $V_0 \in \mathcal{N}_o G$ be the V of 0.4(MK) for the map $\tilde{s}_1$. Then, for some $V_2 \in \mathcal{N}_o E$ and $\delta_2 > 0$, we have $V_2 \times \mathrm{B}(\delta_2) \subseteq V_0$. Putting now $V_1' = V_2 \cap (-x_1 + U) \cap (p^{-1}[0, \delta_2[)$, we get $V_1' \in \mathcal{N}_o E$.

To show that V_1' can be taken as the V in 0.4(MK), let $t \in I\!\!K \setminus \{0\}$ and $h \in U_1'$ be such that $th \in V_1'$. Then $x_1 + th \in U$ and $h \in p^{-1}[0, \delta_1[$. Writing $k = t^{-1}(g(x_1 + t\,h) - g(x_1))$, we have $\|k\| \le |t|^{-1} p(th) = p(h) < \delta_1$. Hence $w = (h, k) \in U_0$. By the fact that $th \in V_1' \subseteq p^{-1}[0, \delta_2[$, we have $\|tk\| = \|g(x_1 + t\,h) - g(x_1)\| \le p(th) < \delta_2$. So $tw = (th, tk) \in V_0$. Hence $t^{-1} r(th) = t^{-1} s_1(th, g(x_1 + th) - g(x_1)) = t^{-1} s_1(th, tk) = t^{-1} s_1(tw) \in W$.

Above we have shown $\tilde{r}$ to be MK–small. To conclude the proof, let $x_1 + h \in U$, and write $k = g(x_1 + h) - g(x_1)$. Then we have

$$\begin{aligned} 0_F &= f(x_1 + h, g(x_1 + h)) = f(x_1 + h, y_1 + k) \\ &= f(z_1) + f'(z_1)(h, k) + s(h, k) = \ell_1(h) + \ell_2(k) + s(h, k). \end{aligned}$$

Hence
$$\begin{aligned} g(x_1 + h) &= g(x_1) + k = g(x_1) - \ell_2^{-1}(\ell_1(h)) - \ell_2^{-1}(s(h, k)) \\ &= g(x_1) - \ell_2^{-1}(\ell_1(h)) + r(h). \end{aligned}$$ □

Our Implicit Function Theorem is a trivial corollary of the following

1.4 Proposition. *Assume $k \in I\!\!N \cup \{\infty\}$. Let $f : E \times F \supseteq \operatorname{dom} f \to F$ be in C^k, with $f(x_0, y_0) = z_0$ and $\partial_2 f(x_0, y_0)$ bijective $F \to F$. Then there exists an open neighbourhood U of x_0 in E, and $\delta > 0$, such that for $V = \mathrm{B}(y_0, \delta)$ and $g = f^{-1}(z_0) \cap (U \times V)$, the relations* (1), … (4) *below hold.*

(1) $U \times V \subseteq \operatorname{dom} f$

(2) $(E, F, g) \in C^k$ *and* $\operatorname{dom} g = U$ *and* $\operatorname{rng} g \subseteq V$

(3) $\partial_2 f(x, g(x)) \in \mathcal{L}is\,(F, F)$ *for* $x \in U$

(4) $g'(x) = -\,(\partial_2 f(x, y))^{-1} \circ (\partial_1 f(x, y))$ *for* $(x, y) \in g$

Proof. Taking the function $w \mapsto f(w) - z_0$ as f in Lemma 1.3, we find U and δ. Then everything claimed is trivial except $\tilde{g} = (E, F, g) \in C^k$. Using induction, we prove this by showing that $\tilde{g} \in C^l$ for $l < k + 1$. For $l = 0$ this is mere continuity of g, which is given by Lemma 1.3.

Assume now $l < k$ and $\tilde{g} \in C^l$. We assert that $\tilde{g} \in C^{l+1}$. By Proposition 0.5 and 1.3(2), $\tilde{g}$ is differentiable. So it suffices to show $D\tilde{g} \in C^l$. Write $G = E \times F$, $K = \mathcal{L}_{eq}(G, F)$, $L = \mathcal{L}_{eq}(E, F)$ and $M = \mathcal{L}_{eq}(F, F)$. Then $D\tilde{g} = (E, L, g')$. By (4), and with $j_1 : E \to G$ and $j_2 : F \to G$ the natural injections, the map $D\tilde{g}$ can be decomposed as

$x \overset{1}{\mapsto} (x,x) = (x,x') \overset{2}{\mapsto} (x, g(x')) = z \overset{3}{\mapsto} f'(z) = \ell \overset{4}{\mapsto} (\ell,\ell) =$

$(\ell,\ell') \overset{5}{\mapsto} (\ell \circ j_1, \ell' \circ j_2) = (u,v) \overset{6}{\mapsto} (u, -v^{-1}) = (u,w) \overset{7}{\mapsto} w \circ u = g'(x)$,

where we have the following maps.

(1) $E \to E \times E$, continuous linear, thus in $C^\infty \subseteq C^l$.

(2) $(\mathrm{id}\, E) \times g : E \times E \supseteq E \times U \to G$, which by the induction hypothesis and by 0.13 is in C^l.

(3) $(G, K, f') \in C^{k-1} \subseteq C^l$.

(4) $K \to K \times K$, continuous linear, thus in $C^\infty \subseteq C^l$.

(5) $\Gamma_1 \times \Gamma_2 : K \times K \to L \times M$, where we have the partial composition functions $\Gamma_1 : K \ni \ell \mapsto \ell \circ j_1 \in L$ and $\Gamma_2 : K \ni \ell' \mapsto \ell' \circ j_2 \in M$. These are continuous linear, hence so is the map (5), thus in $C^\infty \subseteq C^l$.

(6) $(\mathrm{id}\, L) \times (-\mathrm{inv}) : L \times M \supseteq L \times \mathcal{L}is\,(F,F) \to L \times M$, by 0.13 in C^l.

(7) $\mathrm{comp} : L \times M \to L$, in $C^\infty \subseteq C^l$, by what we said after 0.8.

By Proposition 0.11, we have $D\tilde{g} \in C^l$. □

2. Infinite dimensional manifolds. To define manifolds, assume we are given a class C_s consisting of some G–differentiable maps $\tilde{f} = (E, F, f)$. Assume also that C_s becomes a category **S** under the composition introduced after 0.8. We also require the chain rule formula $(g \circ f)'(x)h = g'(f(x))(f'(x)h)$ to hold for $\tilde{f}, \tilde{g} \in C_s, x \in f^{-1}(\mathrm{dom}\, g), h \in E$. Let $\mathcal{O}$ be the class of objects of **S**, and for $E, F \in \mathcal{O}$, let $\mathcal{S}(E,F)$ be the set of all functions f such that $(E,F,f) \in C_s$.

We want to build categories whose objects are manifolds. To make this possible, manifolds, as defined below, have to be sets and not proper classes. To this end, we use the following technical trick. We call a space $E \in \mathcal{O}$ *standard*, and write $E \in \mathcal{O}_s$, iff the underlying vector space of E is of the form $\mathbb{K}^{(I)}$ for some cardinal number I. The vectors of $\mathbb{K}^{(I)}$ are those of $\mathbb{K}^I$ with only finitely many nonzero coordinates. Then, for every $E \in \mathcal{O}$, the class $\{ F : F \in \mathcal{O}_s \text{ and } \mathcal{L}is\,(E,F) \neq \emptyset \}$ is always a set. We assume this set to be nonempty.

Later we choose $\mathcal{O}$ to be the class of all *Fréchet spaces*, and with $k \in \mathbb{N} \cup \{\infty\}$ fixed (so $k > 0$) we let C_s be the class C^k restricted to those maps (E,F,f) for which $E, F \in \mathcal{O}$. To enlarge the applicability of what follows, recall Remarks 0.12.

For any class A, recall that $\mathrm{dom}\, A = \{ x : \exists\, y \;;\; (x,y) \in A \}$, $\mathrm{rng}\, A = \{ y : \exists\, x \;;\; (x,y) \in A \}$, and put $\mathrm{Dom}\, A = \bigcup \{ \mathrm{dom}\, \phi : \phi \in \mathrm{dom}\, A \}$.

2.1 Definitions. A relation A is called an *atlas* iff $\mathrm{Dom}\, A$ is a set, $\emptyset \notin \mathrm{dom}\, A$, and $(\phi, E), (\psi, F) \in A$ implies that ϕ is an injection, that $E, F \in \mathcal{O}$ and that $\psi \circ \phi^{-1} \in \mathcal{S}(E,F)$.

For an atlas A, the members of $\mathrm{rng}\, A$ are its *model spaces*. Any $\alpha = (\phi, E)$, for which also $A \cup \{\alpha\}$ is an atlas and $\mathrm{dom}\, \phi \subseteq \mathrm{Dom}\, A$, is called a *chart* for A. If also $x \in \mathrm{dom}\, \phi$, then α is called a *chart at* x. The *underlying* *set* of an atlas A is Dom A. An atlas is called *standard* iff its model spaces are standard.

A standard atlas M is called a *manifold* iff we have $(\phi, E) \in M$ whenever $M \cup \{(\phi, E)\}$ is a standard atlas with dom $\phi \subseteq$ Dom A. Thus manifolds are maximal standard atlases for some fixed set.

Let M, N be manifolds and f a function with $f \subseteq (\text{Dom } M)\times(\text{Dom } N)$ [1]. Then $f : M \supseteq \text{dom } f \to N$ is called *smooth*[2] iff $\psi \circ f \circ \phi^{-1} \in \mathcal{S}(E, F)$ holds whenever $(\phi, E) \in M$ and $(\psi, F) \in N$. We also say that the (manifold) map, i.e., the triple (M, N, f) is smooth. Note that we *do not* require dom $f =$ Dom M.

Recall that for a topological vector space G, its closed topological vector subspaces E, F are called *pairwise complemented*, and each is called a *topological complement* of the other, iff the function $\ell : E \times F \ni (x, y) \mapsto x + y \in G$ is a linear homeomorphism $E \times F \to G$. By the open mapping theorem, for a *Fréchet space* G, it suffices for ℓ to be bijective.

2.2 Definitions. Let M be a manifold modelled on topological vector spaces and $S \subseteq$ Dom M. Then S is called a *submanifold* of M iff for every $x \in S$, there exists a chart (ϕ, G) for M at x and pairwise complemented topological vector subspaces E, F of G, such that $E \cap (\text{rng } \phi) = \phi(S)$, and such that this set is open in E. We also call the pair (M, S) a submanifold.

The manifold (structure) $\tilde{S}$ of a submanifold (set) S of M is given by the following construction. Let Prop $[M, N, S]$ mean the condition that N is a manifold with Dom $N = S$, and that for all manifolds P and functions $f : \text{Dom } P \to S$ we have the equivalence: (P, M, f) smooth $\Leftrightarrow (P, N, f)$ smooth. Then $\tilde{S} = \{ \alpha : \exists\, N ; \alpha \in N \text{ and Prop } [M, N, S] \}$ is the unique N satisfying condition Prop $[M, N, S]$; cf. [7; pp. 24–25].

2.3 Definitions. The *manifold generated* by an atlas A is the set $\{ (\phi, E) : A \cup \{(\phi, E)\} \text{ is an atlas, dom } \phi \subseteq \text{Dom } A \text{ and } E \in \mathcal{O}_s \}$. If $A \cup B$ is an atlas and Dom $A =$ Dom B, then A, B are *equivalent* atlases and they generate the same manifold.

The *manifold topology* defined by an atlas A is the topology for Dom A generated by the base $\{ \phi^{-1}(\text{dom } f) : \exists\, E, F \in \mathcal{O} ; (\phi, E) \in A \text{ and } f \in \mathcal{S}(E, F) \}$.

We call g a *local diffeomorphism* $E \to F$ *(at x)* iff both (E, F, g) and (F, E, g^{-1}) belong to C_s (and $x \in \text{dom } g$).

Having given our general basic definitions, we now specialize to the case $\mathcal{O} = \{E : E \text{ Fréchet }\}$, $k \in I\!N \cup \{\infty\}$, $C_s = \{(E, F, f) \in C^k : E, F \in \mathcal{O}\}$.

2.4 Proposition. *Let G be a Fréchet space and F a Banach space. Let $(z_0, y_0) \in f \in \mathcal{S}(G, F)$, and let $\ell_1 = f'(z_0) : G \to F$ be surjective,*

[1] Abusively, $f \subseteq M \times N$

[2] or a C_s– map ; likewise we would say : C_s– atlas / (sub)manifold / diffeomorphism

with Ker ℓ_1 *complemented in* G. *Then there exists a Fréchet space* E *and a local diffeomorphism* $g : E\times F \to G$ *at* $(0_E, y_0)$, *satisfying* $g(0_E, y_0) = z_0$, $\operatorname{rng} g \subseteq \operatorname{dom} f$, *and such that* $(f \circ g)(x, y) = y$ *holds for* $(x, y) \in \operatorname{dom} g$.

Proof. Let E be Ker ℓ_1 considered as a topological vector subspace of G. Then E is Fréchet. Let F_1 be a complement of E. Then also F_1 is Fréchet. Now $\ell_1|F_1$ is a continuous linear isomorphism $F_1 \to F$. By the open mapping theorem, also $\ell = (\ell_1|F_1)^{-1}$ is continuous. (So F_1 is normable, hence Banach.) Letting $p : G \to E$ and $p_1 : G \to F_1$ be the projections, we have $\ell \circ \ell_1 = p_1$. Write $x_0 = z_0 - \ell(y_0)$.

Put $E_1 = E \times F$ and $G_1 = E_1 \times F$. Consider $\tilde{f}_1 = (G_1, F, f_1)$, where $f_1 : ((x, v), y) \mapsto f(x + x_0 + \ell(y)) - v$ for $x + x_0 + \ell(y) \in \operatorname{dom} f$. Then for $w_0 = (0_E, y_0)$, our map $\tilde{f}_1$ satisfies the requirements of the Implicit Function Theorem, with $f_1(w_0, y_0) = 0_F$ and $\partial_2 f_1(w_0, y_0) = f'(z_0) \circ \ell = \operatorname{id} F$. So we find open neighbourhoods U_1 and V_1, of w_0 and y_0, in the spaces E_1 and F, respectively, and a function $g_1 : U_1 \to V_1$, satisfying $(E_1, F, g_1) \in C_s$ and $g_1(0_E, y_0) = y_0$ and $f(x + x_0 + \ell(g_1(x, v))) = v$ for $(x, v) \in U_1$. Differentiation with respect to v yields $\partial_2 g_1(0_E, y_0) = \operatorname{id} F$.

Let $g_0 : U_1 \ni (x, y) \mapsto x + x_0 + \ell(g_1(x, y))$. Then $(E_1, G, g_0) \in C_s$ and

$$(*) \qquad g_0(0_E, y_0) = z_0 \text{ and } (f \circ g_0)(x, y) = y \text{ for } (x, y) \in U_1 .$$

Taking into account the direct sum decomposition of G, we can check that g_0 is injective. However, we do not know whether $(G, E_1, g_0^{-1}) \in C_s$. To prove that a suitable restriction of g_0 is a local diffeomorphism, we make another application of the Implicit Function Theorem.

Let $G_2 = G \times F$, and consider the map $\tilde{f}_2 = (G_2, F, f_2)$, where $f_2 : (z, y) \mapsto g_1(p(z-z_0), y) - \ell_1(z-z_0)$ for $(p(z-z_0), y) \in U_1$. Then $\tilde{f}_2$ satisfies the requirements of the Implicit Function Theorem, with $f_2(z_0, y_0) = y_0$ and $\partial_2 f_2(z_0, y_0) = \operatorname{id} F$. So we find open neighbourhoods U_2 and V_2, of z_0 and y_0, in the spaces G and F, respectively, and a function $g_2 : U_2 \to V_2$ satisfying $(G, F, g_2) \in C_s$ and $g_1(p(z - z_0), g_2(z)) = y_0 + \ell_1(z - z_0)$ for all $z \in U_2$. We also have $g_2(z_0) = y_0$.

Let $h : U_2 \ni z \mapsto (p(z-z_0), g_2(z)) \in E_1$, and put $g = g_0|(\operatorname{rng} h)$. Then $(G, E_1, h) \in C_s$ and $\operatorname{rng} h \subseteq \operatorname{dom} g_0$. In order to show $(E_1, G, g) \in C_s$, it suffices to show that rng h is open in E_1. For all $z \in U_2$, we have

$$\begin{aligned} g_0(h(z)) &= g_0(p(z - z_0), g_2(z)) \\ &= p(z - z_0) + x_0 + \ell(g_1(p(z - z_0), g_2(z))) \\ &= p(z) - p(z_0) + z_0 - \ell(y_0) + \ell(y_0) + \ell(\ell_1(z - z_0)) \\ &= p(z) - p(z_0) + z_0 + p_1(z) - p_1(z_0) = z. \end{aligned}$$

Hence $h \subseteq g_0^{-1}$. Then $\operatorname{rng} h = g_0^{-1}(\operatorname{dom} h) = g_0^{-1}(U_2)$ is open in E_1, because g_0 is continuous. From $h \subseteq g_0^{-1}$ and $g = g_0|(\operatorname{rng} h)$ it follows that $g^{-1} = h$. From $(*)$ and $(0_E, y_0) = h(z_0) \in \operatorname{rng} h$, we see that g satisfies the remaining assertions. □

In Banach space calculus there is a corresponding "dual" theorem to

Proposition 2.4 above; cf. [7; p. 16, Corollary 1s] or [3; p. 215, A.9.]. Here we cannot prove such a theorem, because we would need an inverse function theorem for maps between general Fréchet spaces.

We now prove our Theorem stated at the beginning. Tangent spaces and tangent maps are defined as in [7; pp. 26–27].

Proof. For given $x \in f^{-1}(S)$, it suffices to find a chart (ϕ, E) for M at x, such that for some pairwise complemented topological vector subspaces E' and E'' of E, the set $\phi(f^{-1}(S))$ is included and open in E'. To prove this, choose charts $(\phi_1, G) \in M$ at x and $(\psi, F) \in N$ at $y = f(x)$, such that for some pairwise complemented Banach subspaces F_1 and E_2 of F, we have $F_1 \cap (\operatorname{rng} \psi) = \psi(S)$ and such that this set is open in F_1. Let q be the projection $F \to E_2$, and put $\varphi = q \circ \psi \circ f \circ \phi_1^{-1}$. Then we have the commutative diagram

$$\begin{array}{ccccc} T_xM & \xrightarrow{\;Tf_x\;} & T_yN & \xrightarrow{\;\pi\;} & T_yN/T_yS \\ \downarrow\cong & & \cong\downarrow & & \cong\downarrow \\ G & \xrightarrow[(\psi\circ f\circ\phi_1^{-1})'(\phi_1(x))]{} & F & \xrightarrow[q]{} & E_2 \end{array}$$

where $\cong$ means linear homeomorphism. To apply Proposition 2.4, note that $\varphi'(\phi_1(x)) = q \circ ((\psi \circ f \circ \phi_1^{-1})'(\phi_1(x)))$, and use conditions (1) and (2) of our Theorem. So we find a Fréchet space E_1, and a local diffeomorphism $\theta : E = E_1 \times E_2 \to G$, such that $\theta(0_{E_1}, q(\psi(y))) = \phi_1(x)$, and $\varphi(\theta(u, v)) = v$ holds for $(u, v) \in \operatorname{dom} \theta$. Writing $\phi = (\theta^{-1} \circ \phi_1)|(f^{-1}(\operatorname{dom} \psi))$, we get a chart (ϕ, E) for M at x. Moreover, we have $\phi(f^{-1}(S)) =$

$$\begin{aligned} &= \theta^{-1}(\phi_1(f^{-1}(S) \cap f^{-1}(\operatorname{dom} \psi))) \\ &= \theta^{-1}(\phi_1(f^{-1}(S \cap (\operatorname{dom} \psi)))) \\ &= \theta^{-1}(\phi_1(f^{-1}(\psi^{-1}(\psi(S))))) \\ &= \theta^{-1}(\phi_1(f^{-1}(\psi^{-1}(F_1 \cap (\operatorname{rng} \psi))))) \\ &= \theta^{-1}(\phi_1(f^{-1}(\psi^{-1}(q^{-1}(0_F))))) = (\varphi \circ \theta)^{-1}(0_F) \end{aligned}$$

$= (E_1 \times \{0_F\}) \cap (\operatorname{dom} \theta)$, which is open in the topological vector subspace $E' = E_1 \times \{0_F\}$ of E. □

2.5 Corollary. *Let $k \in \mathbb{N} \cup \{\infty\}$, and let $f : E \supseteq \operatorname{dom} f \to F$ be in C^k, with E Fréchet and F Banach. Let $y \in \operatorname{rng} f$, and let ϕ be the identity either on* dom f *or on E. Then, under conditions* (1) *and* (2) *below, $f^{-1}(y)$ is a submanifold of the manifold generated by the atlas $\{(\phi, E)\}$. Condition* (2) *is superfluous if F is finite dimensional.*

(1) $f'(x) : E \to F$ *is surjective for all* $x \in f^{-1}(y)$

(2) $\operatorname{Ker} f'(x)$ *is complemented in E for all* $x \in f^{-1}(y)$

Under the conditions of Corollary 2.5, one can prove that the manifold topology and the induced subspace topology of $f^{-1}(y)$ coincide. One can also see that the submanifold has an equivalent atlas modelled on closed subspaces of E having complements linearly homeomorphic to F.

2.6 Remark. The proofs we have given above, allow, with only minor modifications, the following immediate generalizations of our results.

(1) In Proposition 2.4 and Corollary 2.5, if we assume F to be finite dimensional, the word "Frechet" can be replaced by "locally convex". Moreover, in our Theorem, if we require T_yN/T_yS to be finite dimensional, then the manifolds M and N can be modelled on arbitrary locally convex spaces. Furthermore, the assumptions on the complementedness of the kernels are superfluous.

(2) Let us call a locally convex space E an *omB-space* (for open mapping Banach) iff for all Banach spaces F and all linear surjections $\ell : E \to F$, we have the implication: ℓ continuous $\Rightarrow$ ℓ open. Then, in Proposition 2.4 and Corollary 2.5, the word "Frechet" can be replaced by "omB-". Moreover, in our Theorem, the manifold M only have to be modelled on omB-spaces and N on arbitrary locally convex spaces, if we require T_yN/T_yS to be Banach.

For example, all (locally convex) *webbed* spaces (see [J; pp. 89–93]) are omB-spaces. Concrete examples of webbed spaces are all Frechet spaces, the test function spaces $\mathcal{D}(\Omega)$, and the distribution spaces $\mathcal{D}'(\Omega)$ and $\mathcal{S}'(\mathbb{R}^d)$.

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